Robust surface intersection algorithms and trimmed surface representations

Rida Farouki

Mechanical Engineering, U. C. Davis

Joel Hass

Mathematics, U. C. Davis

Tom Sederberg

Computer Science, Brigham Young University

Other Participants

- Students
  - Junzhe Miao, Ph.D.
  - Ryan Petrie, Ph.D.
  - Michael Smith, MS
- Post-Docs
  - Jianmin Zheng, Xiaowen Song
- Collaborators
  - Chee Yap (NYU, exact geometry computation)
  - David Cox (Amherst, algebraic geometry)

- Tom Sederberg — 45 minutes
- Chee Yap — 45 minutes

- I-TANGO (Tom Peters, et. al.)
- I-WALTZ (Rida Farouki, et. al.)

SIAM Workshop on Integration of CAD and CFD (UC Davis, 1999)

Finite Element Meshes for CFD from CAD models

Typical CFD analysis (aircraft, 50 million elements):

- 10-20 minutes for surface meshing
- 3-4 hours for volume meshing
- 1 hour for actual flow analysis
- 2—4 weeks for “geometry repair”

Workshop on Mathematical Foundations of CAD (MSRI, 1999)

"The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms."

——Consensus opinion

Costs $1 Billion/year

—Tom Peters

Workshop on Mathematical Foundations of CAD (MSRI, 1999)

Piecewise Algebraic Surfaces, Sederberg 1984

#1 greatest example of failed expectations in the history of CAGD

Division of Labor

- Computational Topology
  - UC Davis Mathematics Department
    - strong low-dimensional topology
    - keen interest in computational topology
  - Joel Hass
  - Jesus De Loera
  - Abigail Thompson
  - Bill Thurston
  - Dmitry Fuchs
  - Mikhail Khovanov
  - Gregory Kuperberg

Division of Labor

- Computational Topology
- Homotopy Methods
  - Rida Farouki and Junzhe Miao

Division of Labor

- Computational Topology
- Homotopy Methods
- Exact geometry computation
  - Chee Yap

Division of Labor

- Computational Topology
- Homotopy Methods
- Exact geometry computation
- Computational Algebraic Geometry
  - Tom Sederberg, Jianmin Zheng, Eng-Wee Chionh, David Cox

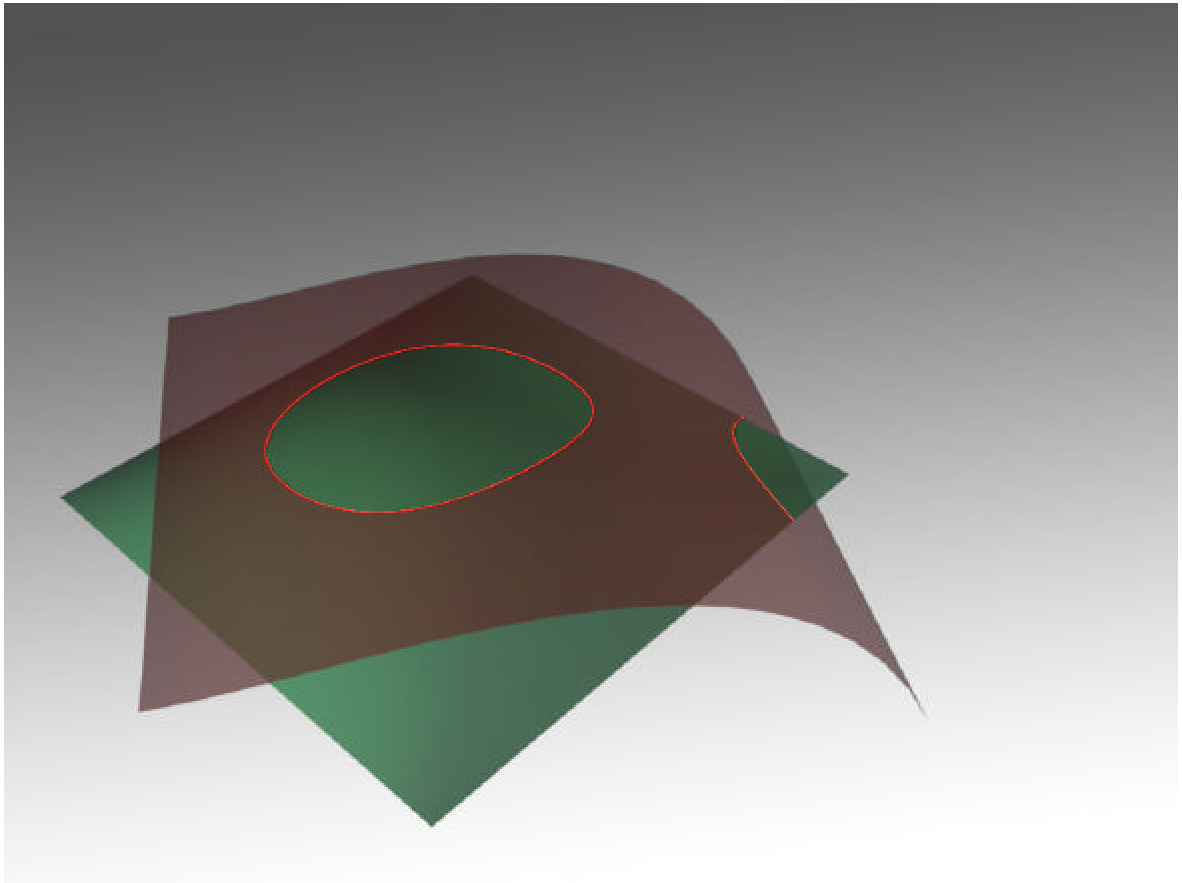
Division of Labor

- Computational Topology
- Homotopy Methods
- Exact geometry computation
- Computational Algebraic Geometry
- Global Differential Geometry
  - Robustly find all components
  - Gauss maps
  - Tom Sederberg, Xiaowen Song

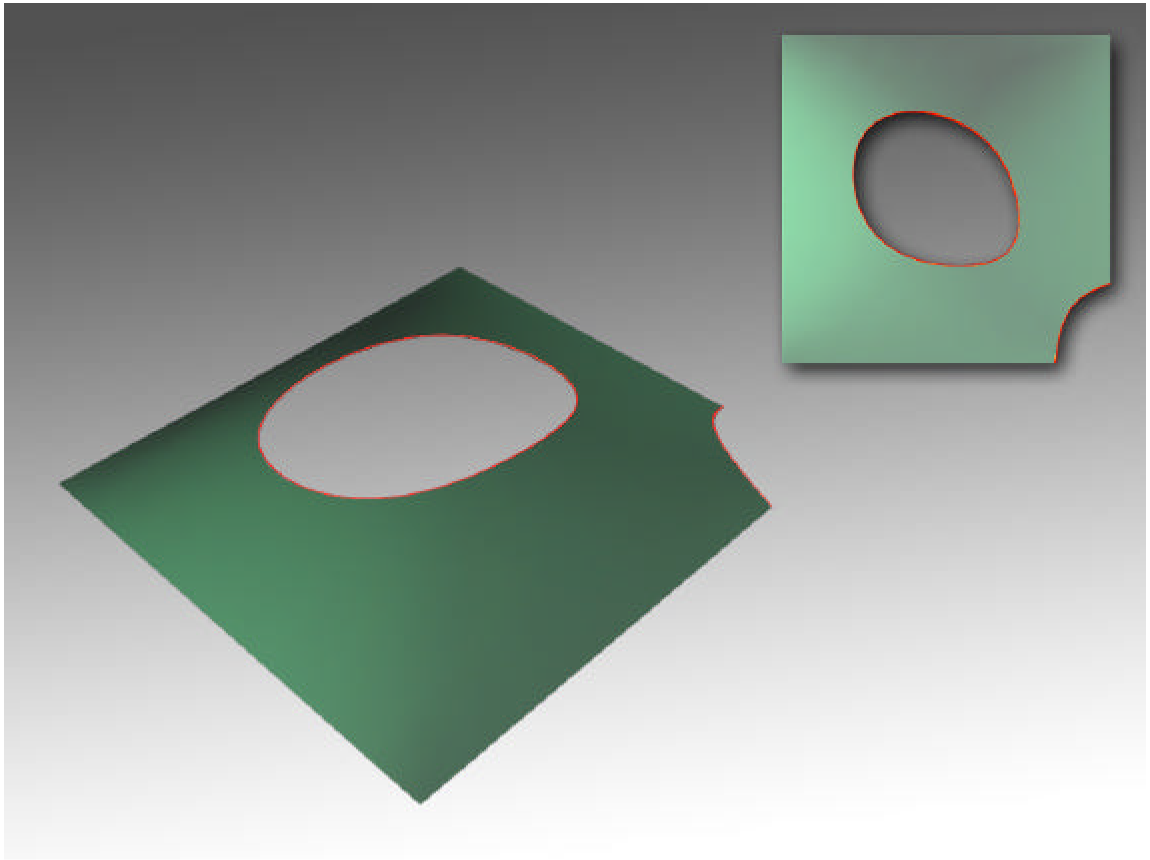
Division of Labor

- Computational Topology
- Homotopy Methods
- Exact geometry computation
- Computational Algebraic Geometry
- Global Differential Geometry
- T-spline representation
  - Tom Sederberg, Jianmin Zheng, Xiaowen Song

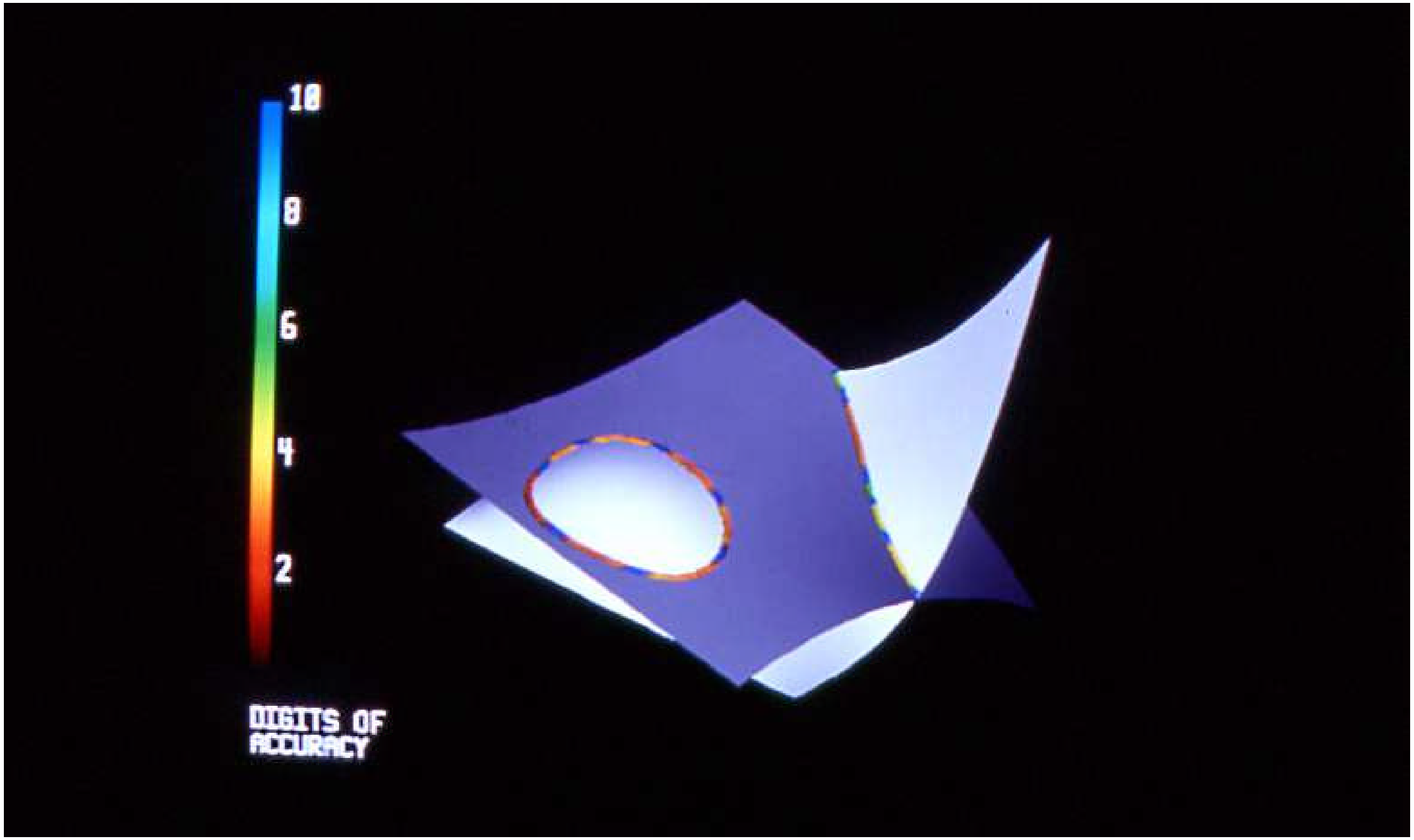
Bi-cubic patch intersection



Trimming Curve



The Gap Problem



How many parametric curves of degree 2 are there?

$$x = \frac{a_2 t^2 + a_1 t + a_0}{c_2 t^2 + c_1 t + c_0}; \quad y = \frac{b_2 t^2 + b_1 t + b_0}{c_2 t^2 + c_1 t + c_0}$$

But you can reparameterize the curve: $t = \dfrac{d_0 u + d_1}{d_2 u + d_3}$

Total dimension is 9-4=5.

How many implicit curves of degree 2 are there?

$$a_0 x^2 + a_1 xy + a_2 y^2 + a_3 x + a_4 y + a_5 = 0$$

But you can assign any coefficient to be 1, so

Dimension is 6-1=5.

How many parametric curves of degree 3 are there?

$$x = \frac{a_3 t^3 + a_2 t^2 + a_1 t + a_0}{c_3 t^3 + c_2 t^2 + c_1 t + c_0}; \quad y = \frac{b_3 t^3 + b_2 t^2 + b_1 t + b_0}{c_3 t^3 + c_2 t^2 + c_1 t + c_0}$$

But you can reparameterize the curve: $t = \dfrac{d_0 u + d_1}{d_2 u + d_3}$

Total dimension is 12-4=8.

How many implicit curves of degree 3 are there?

$$a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_4x^2 + a_5xy + a_6y^2 + a_7x + a_8y + a_9 = 0$$

But you can assign any coefficient to be 1, so

Dimension is 10-1=9.

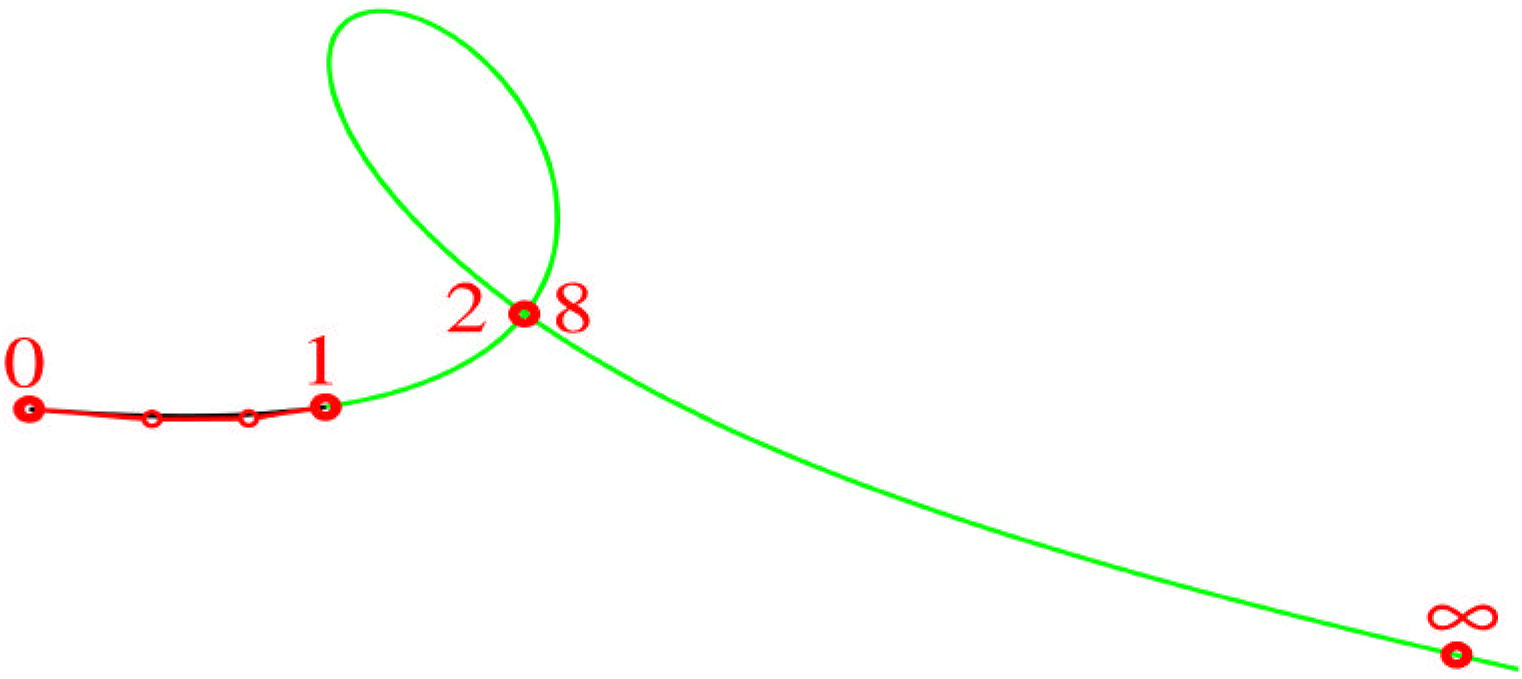
Dimension of the space of planar curves:

Degree	Parametric	Implicit
1	2	2
2	5	5
3	8	9
4	11	14
5	14	20
n	3n-1	n(n+3)/2

Dimension of space of 3D surfaces:

Parametric Degree	Parametric Dimension	Implicit Degree	Implicit Dimension
1X1	8	2	9
2X2	28	8	164
3X3	56	18	1329
4X4	92	32	6544
5X5	136	50	23,426
nXn	4(n+1)^2-8	d=2n^2	(d+1)(d+2)(d+3)/6-1

Cubic Bezier Curve

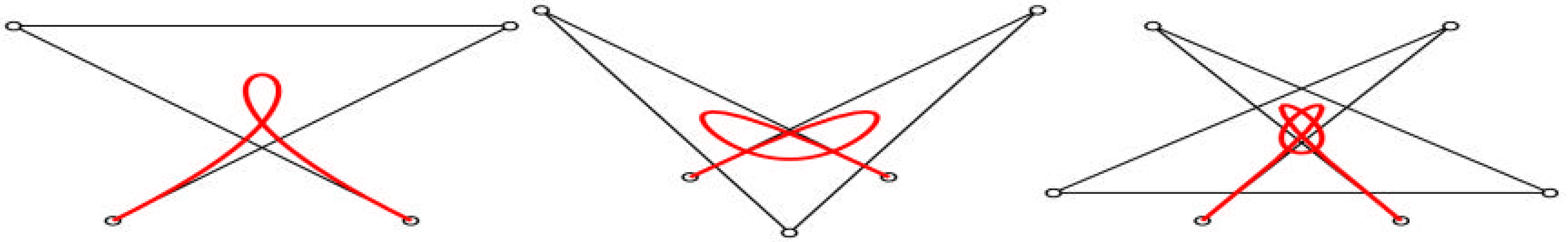


Double Points

A degree n rational curve has

$$\frac{(n-1)(n-2)}{2}$$

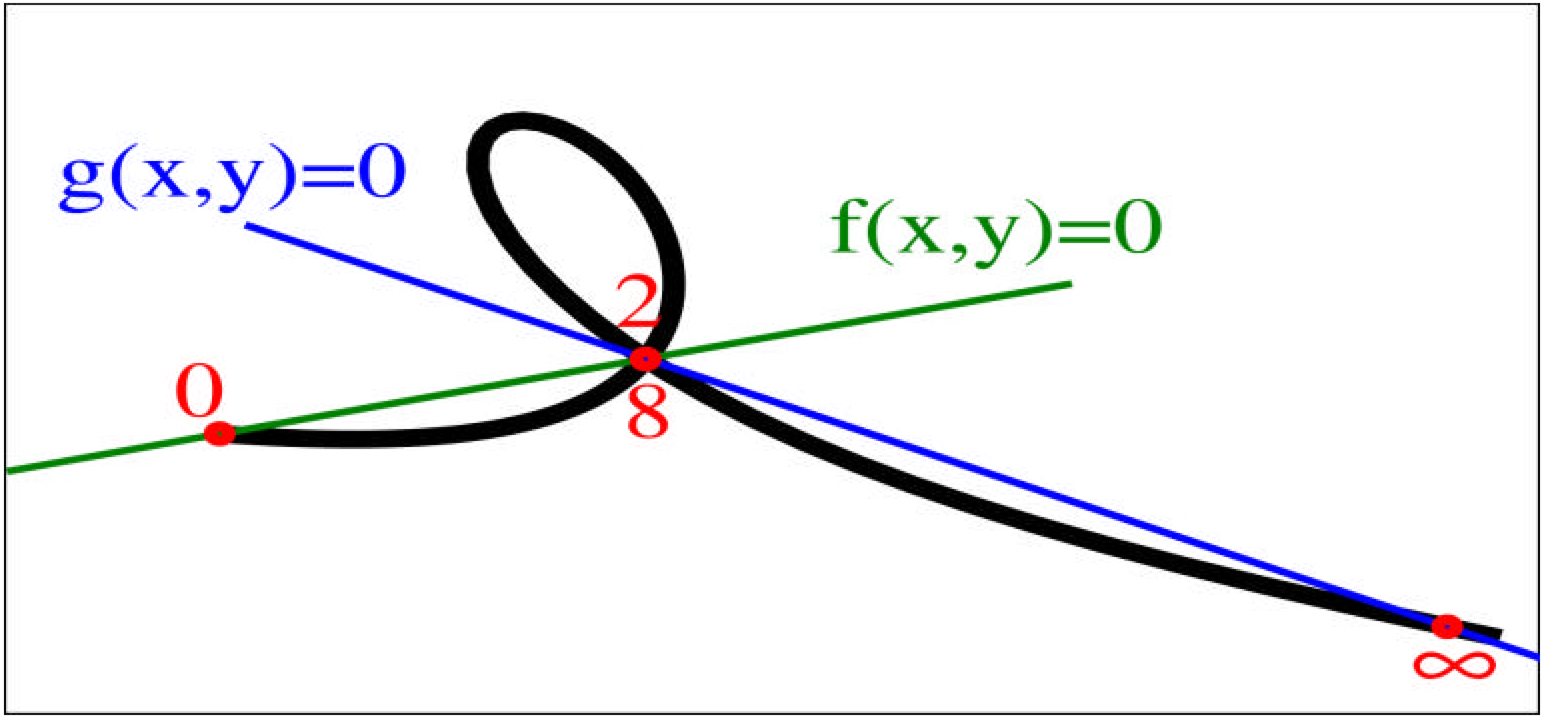
double points.



Dimension of space of planar curves:

Degree	Parametric	Implicit	Double Points
1	2	2	0
2	5	5	0
3	8	9	1
4	11	14	3
5	14	20	6
n	3n-1	n(n+3)/2	(n-1)(n-2)/2

Simplified Implicit Equation

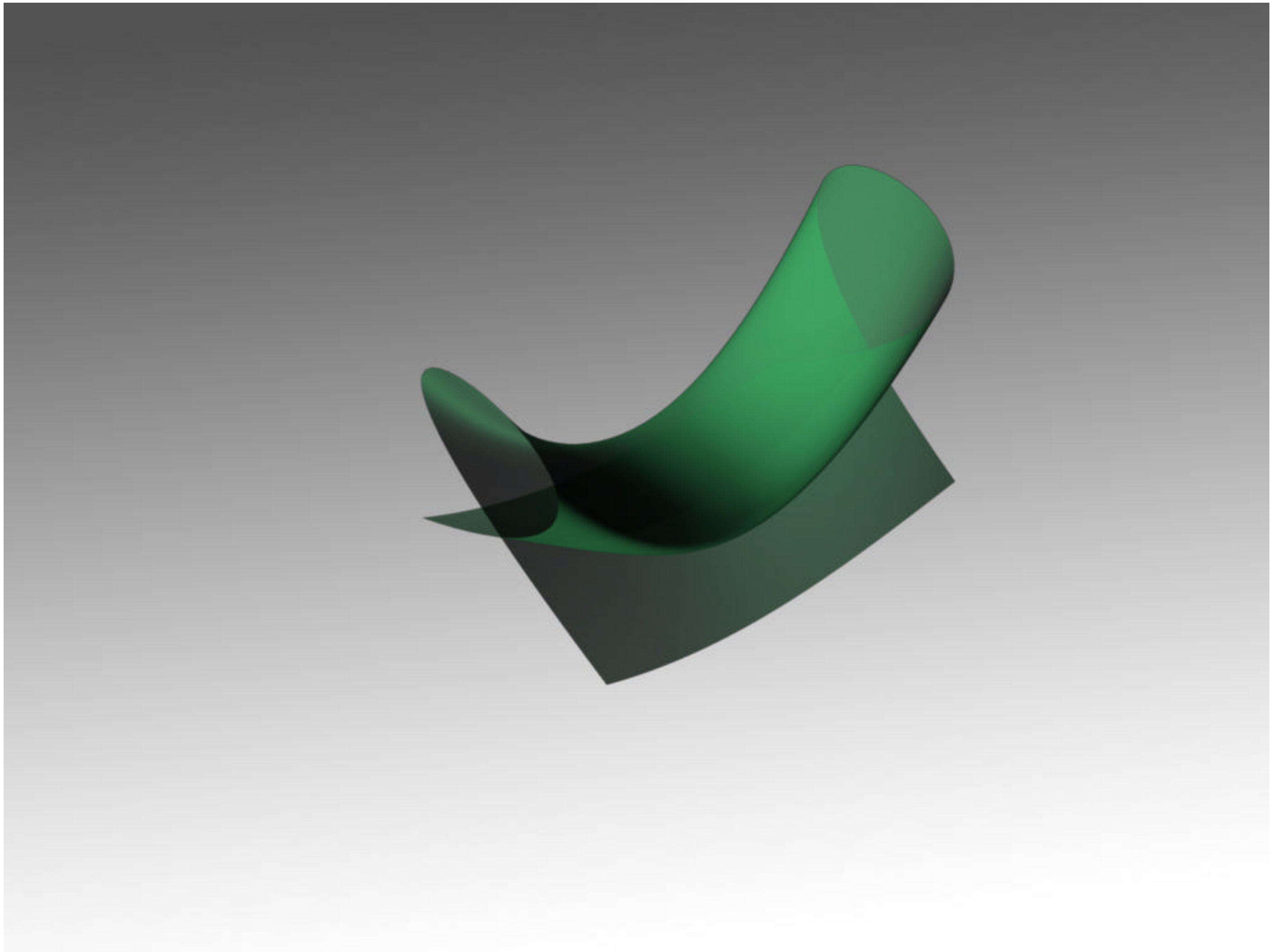


$$f^3 + g^3 + (ax + by + c)fg = 0$$

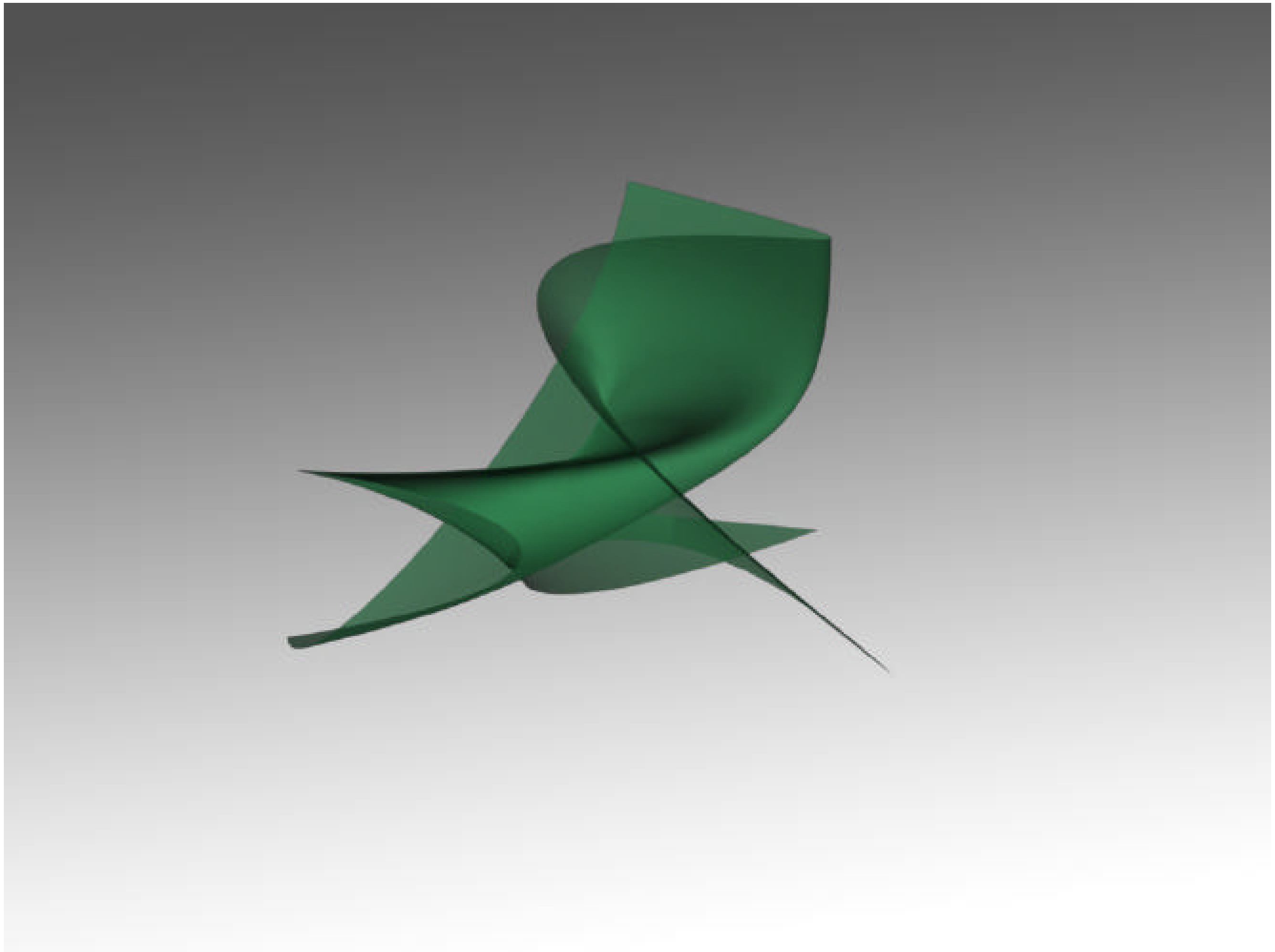
Dimension of space of 3D surfaces:

Parametric Degree	Parametric Dimension	Implicit Degree	Implicit Dimension
1X1	8	2	9
2X2	28	8	164
3X3	56	18	1329
4X4	92	32	6544
5X5	136	50	23,426
nXn	4(n+1)^2-8	d=2n^2	(d+1)(d+2)(d+3)/6-1

Self-Intersection of Bicubic Patch



Self-Intersection of Bicubic Patch

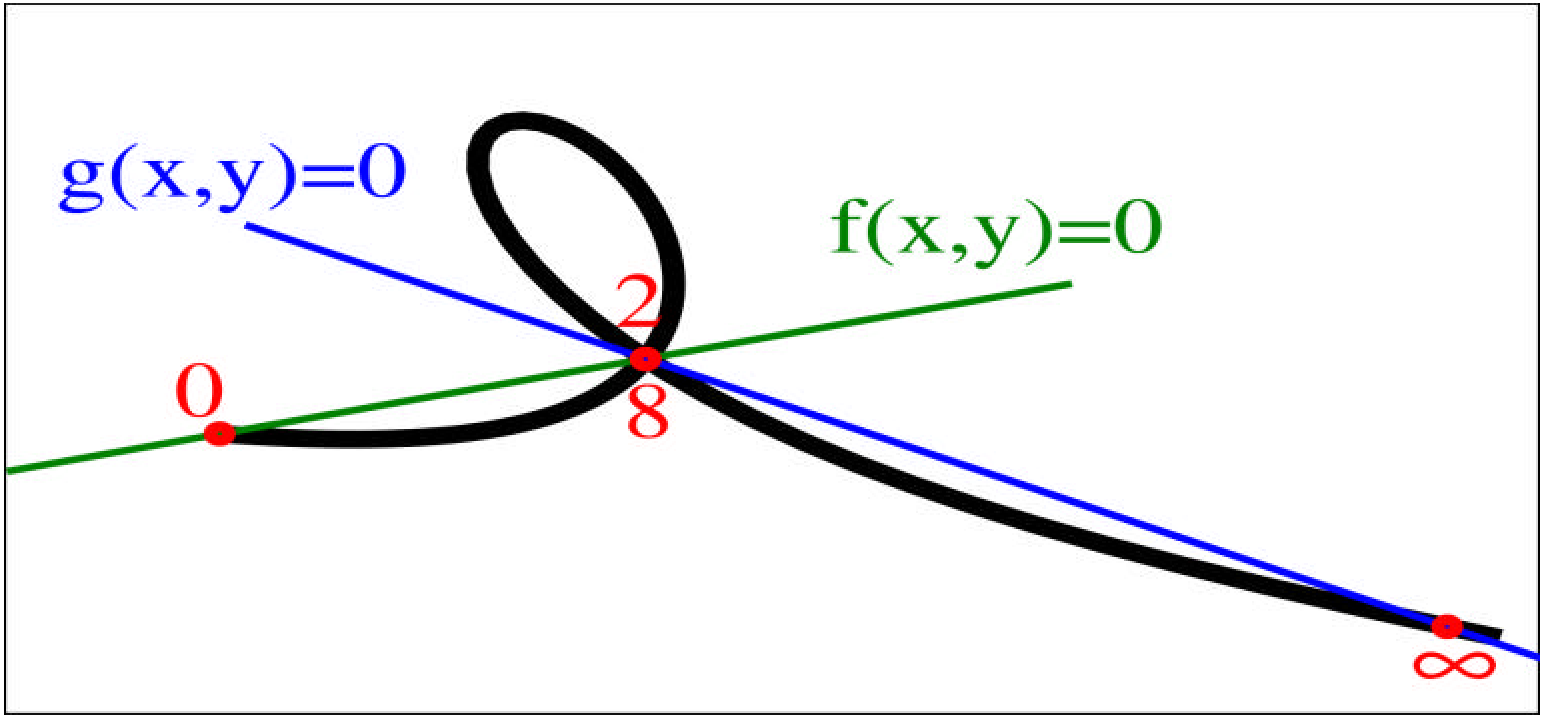


Problem 1.

Can you write the implicit equation of a bi-cubic patch in a simpler form; one that involves fewer (56?) coefficients?

Possible approach: Find a Grobner basis of the singular locus of the patch: I=<F1, F2, …, Fn> and write the implicit equation as a quadratic form in terms of the Grobner basis polynomials.

Simplified Implicit Equation



$$f^3 + g^3 + (ax + by + c)\, fg = 0$$

Problem 2:
Make the method of moving surfaces robust

- Surface implicitization using resultants fails in the presence of base points.
- Method of moving surfaces seems to always work, but no proof (plus it is slow).
- How? Extend to surfaces the idea of a mu-basis for curves.

Problem 3:
Find a starting point on each component of the intersection curve.

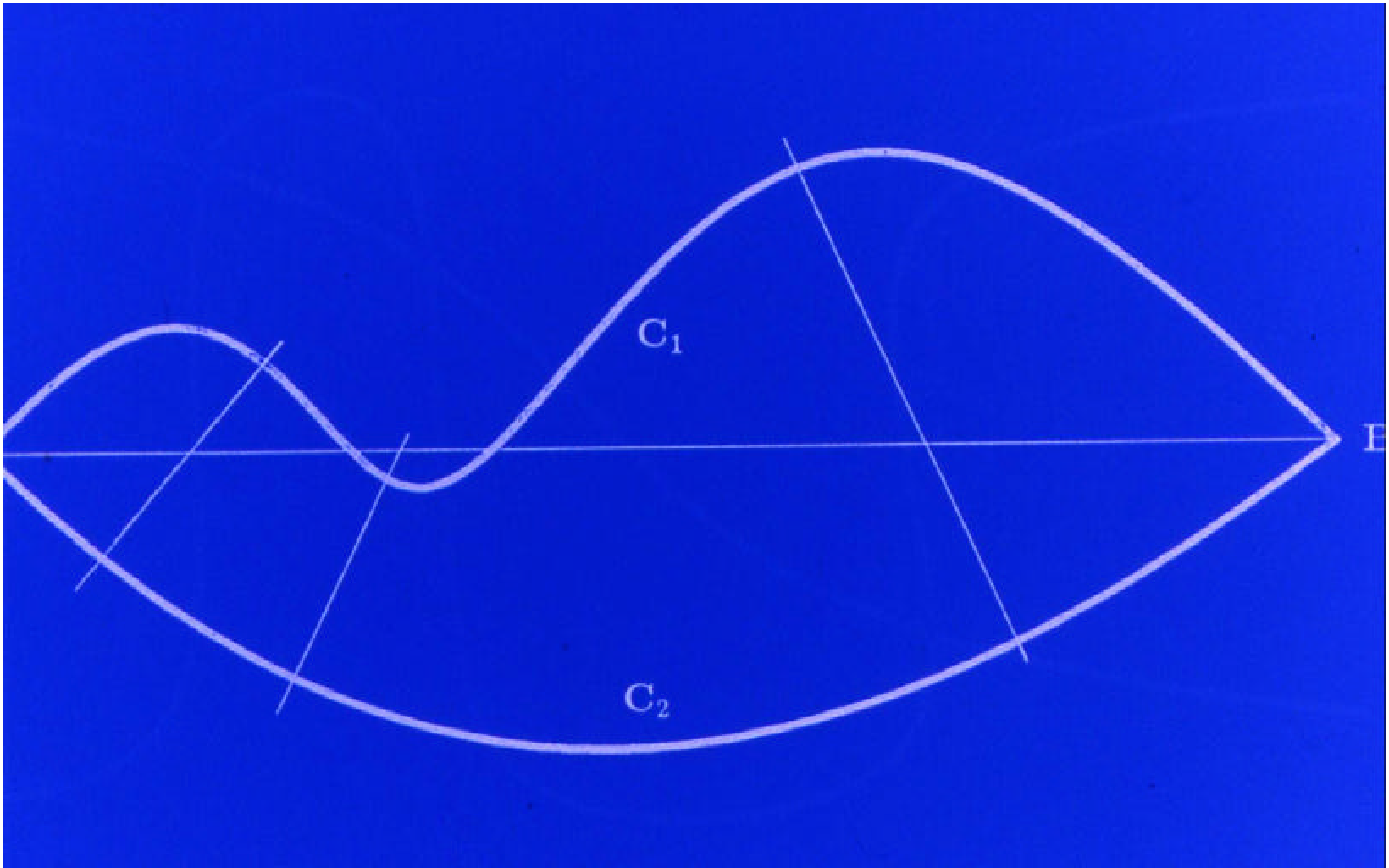
- Collinear normal theorem
- Bezier clipping

Collinear Normal Theorem

- If two planar curves intersect twice (and don’t turn more than 90 degrees) there exists a line which is perpendicular to both curves.

  —Sederberg, Katz, Christiansen 1988

Collinear Normal Theorem

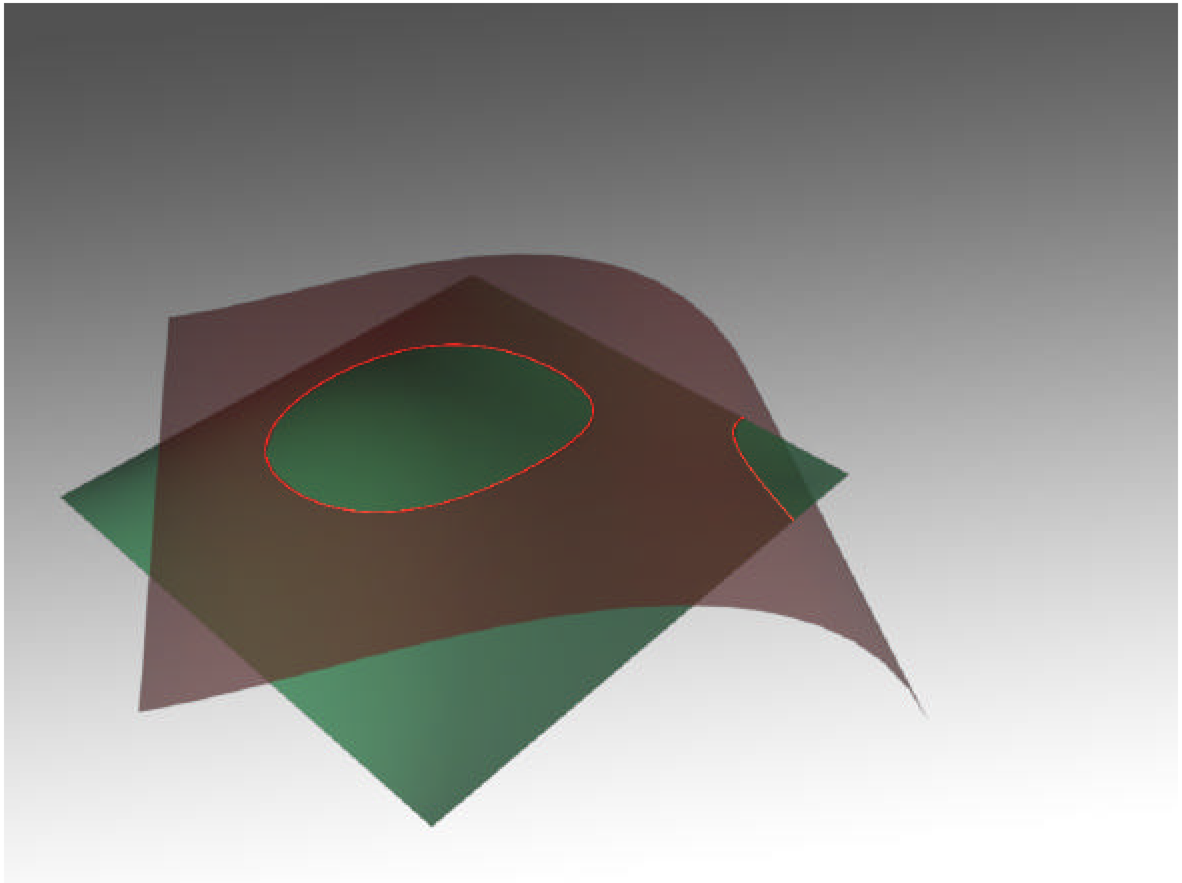


Collinear Normal Theorem

- If two surfaces intersect in a closed loop (and no two normals to a patch are orthogonal) there exists a line which is perpendicular to both surfaces.

—Sederberg, Katz, Christiansen 1988

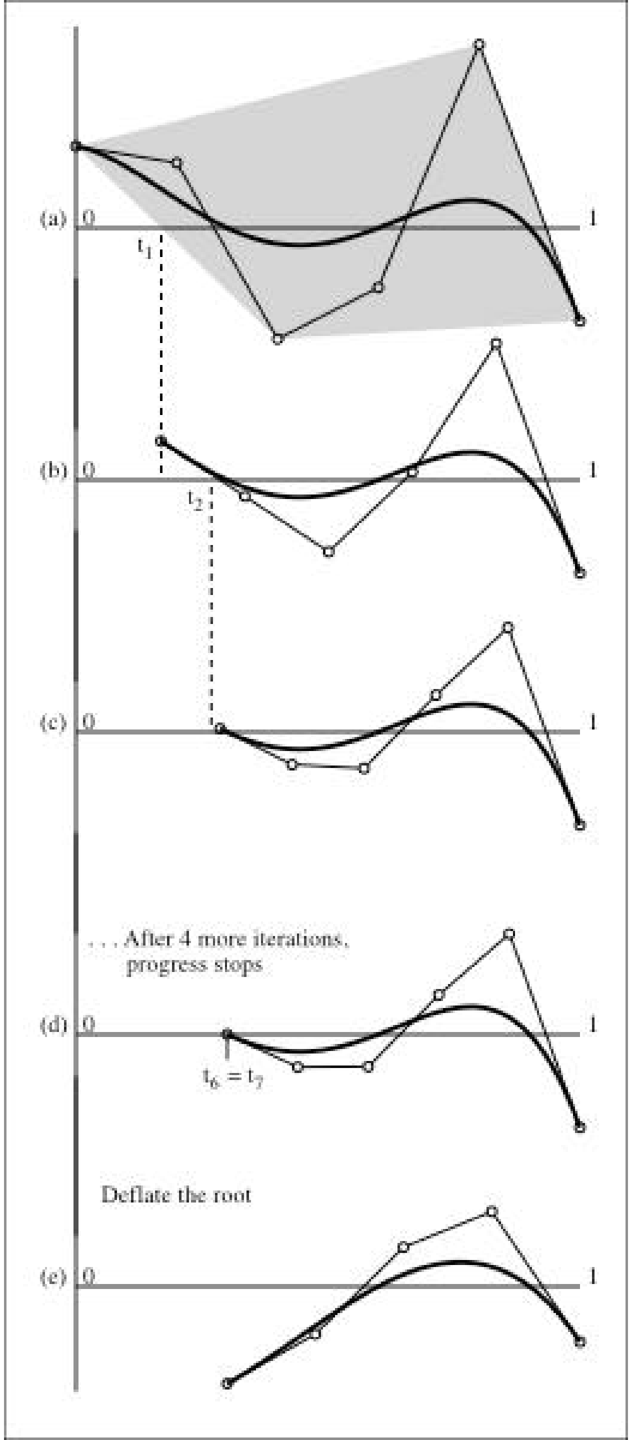
Loop Detection



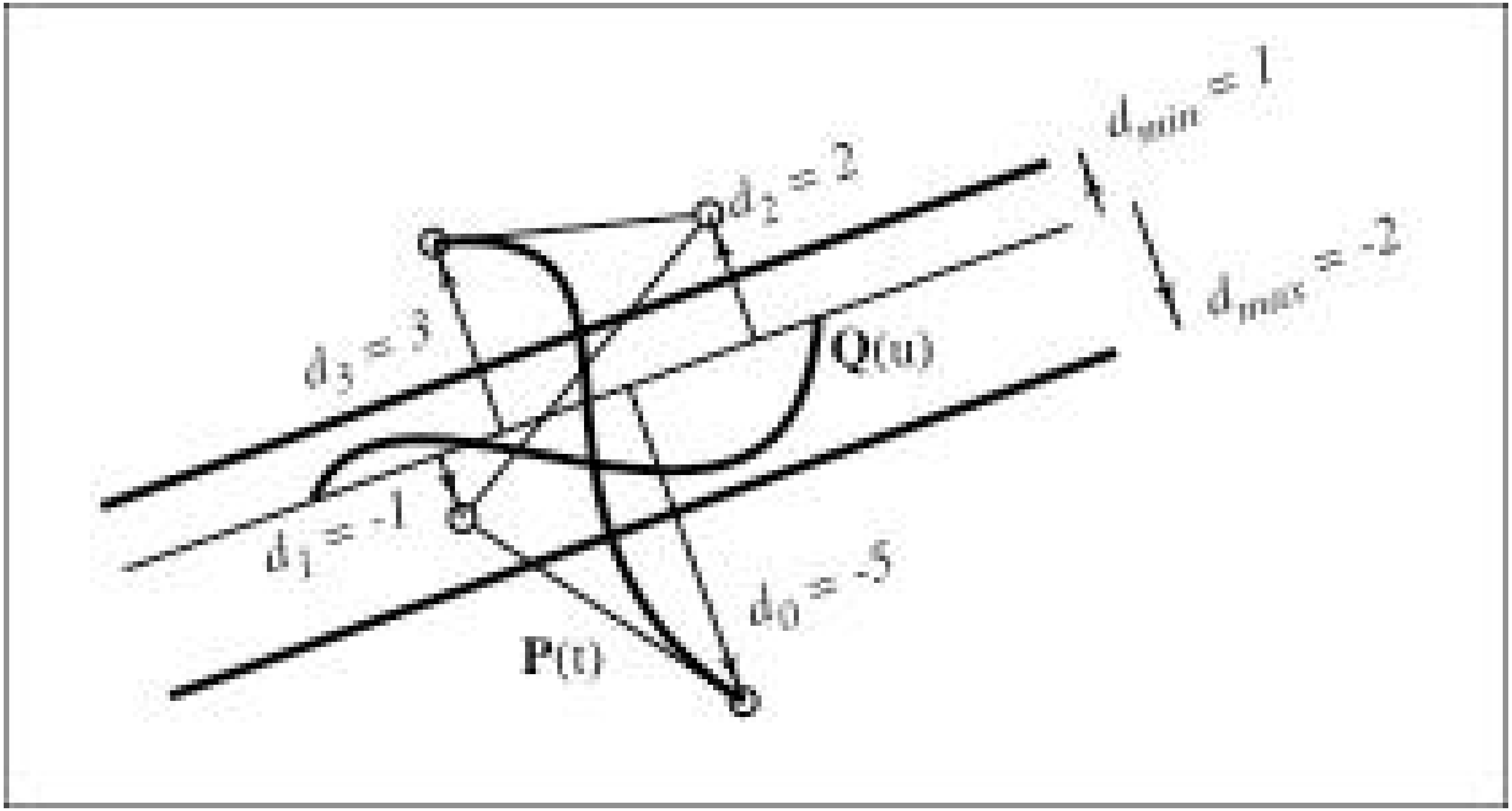
Bezier Clipping

- A method for robustly finding all real zeros of systems of polynomial equations, over a given domain.

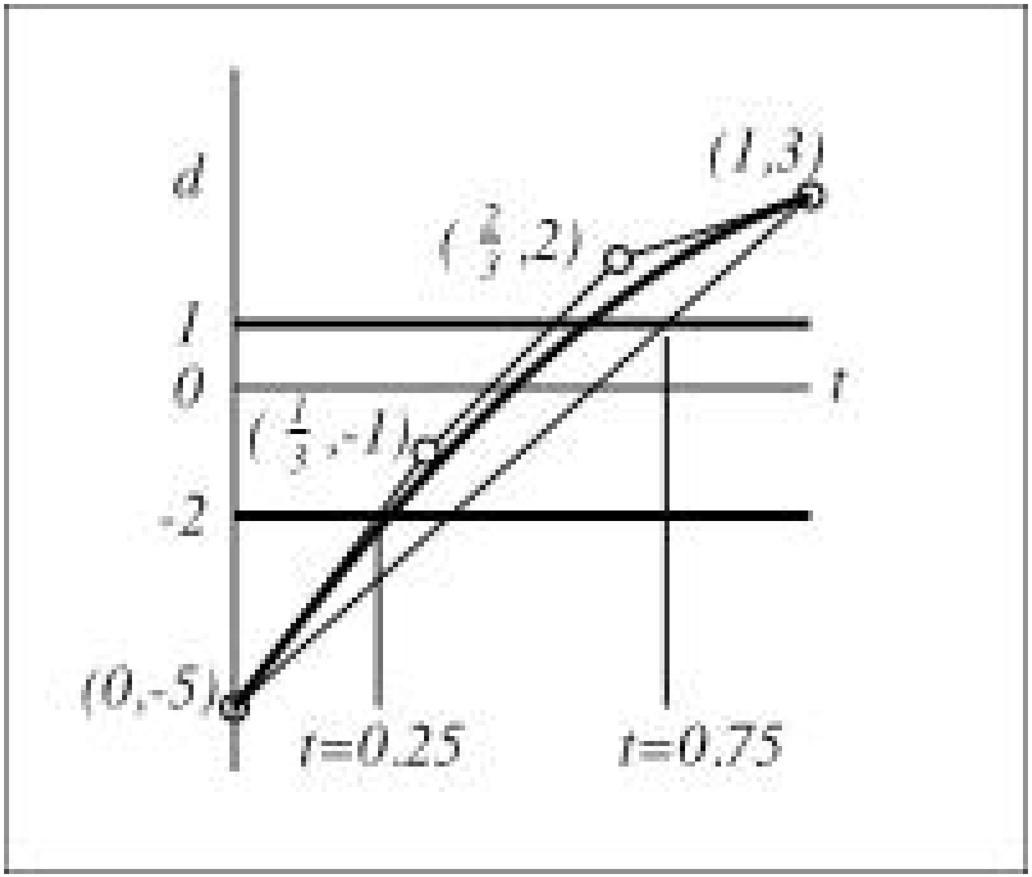
Polynomial Root Finding in Bernstein Form



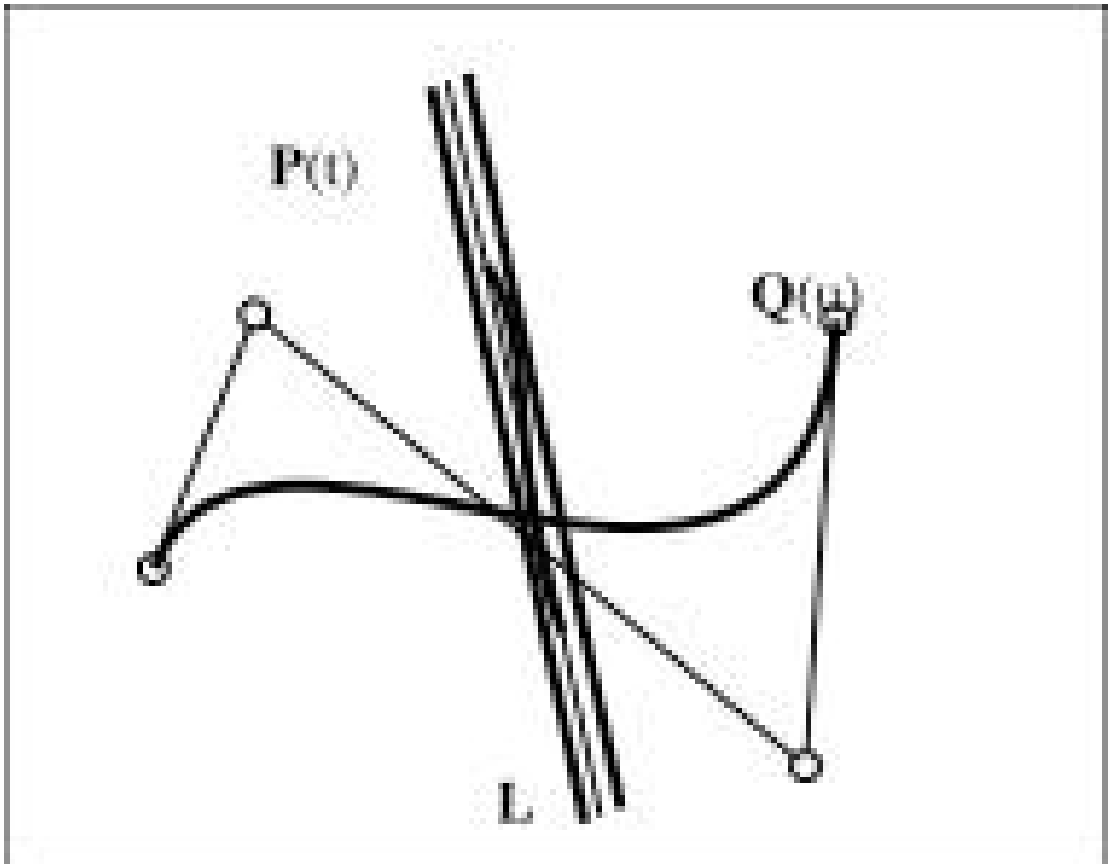
Curve Intersection using Bezier Clipping



Curve Intersection using Bezier Clipping



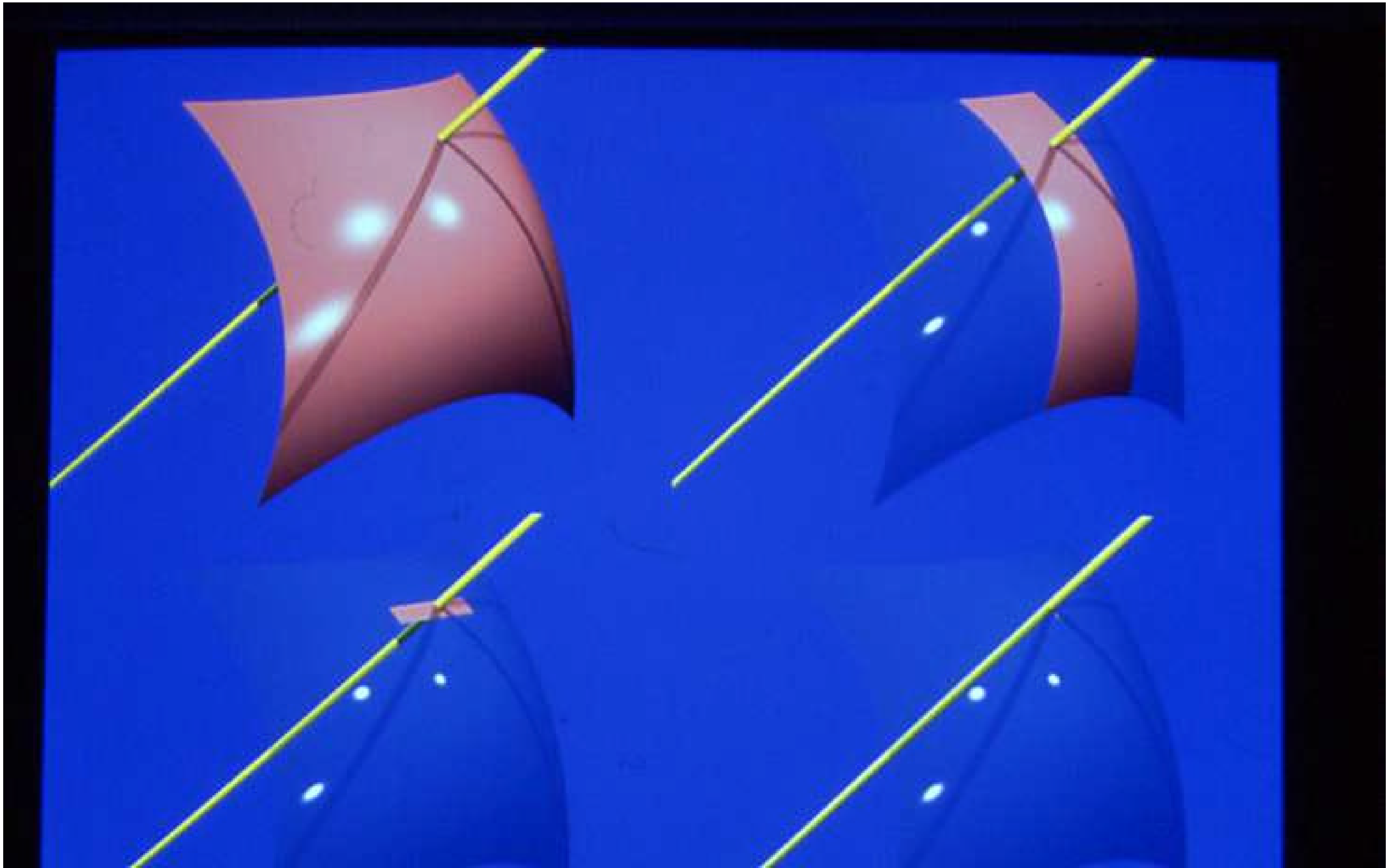
Curve Intersection using Bezier Clipping



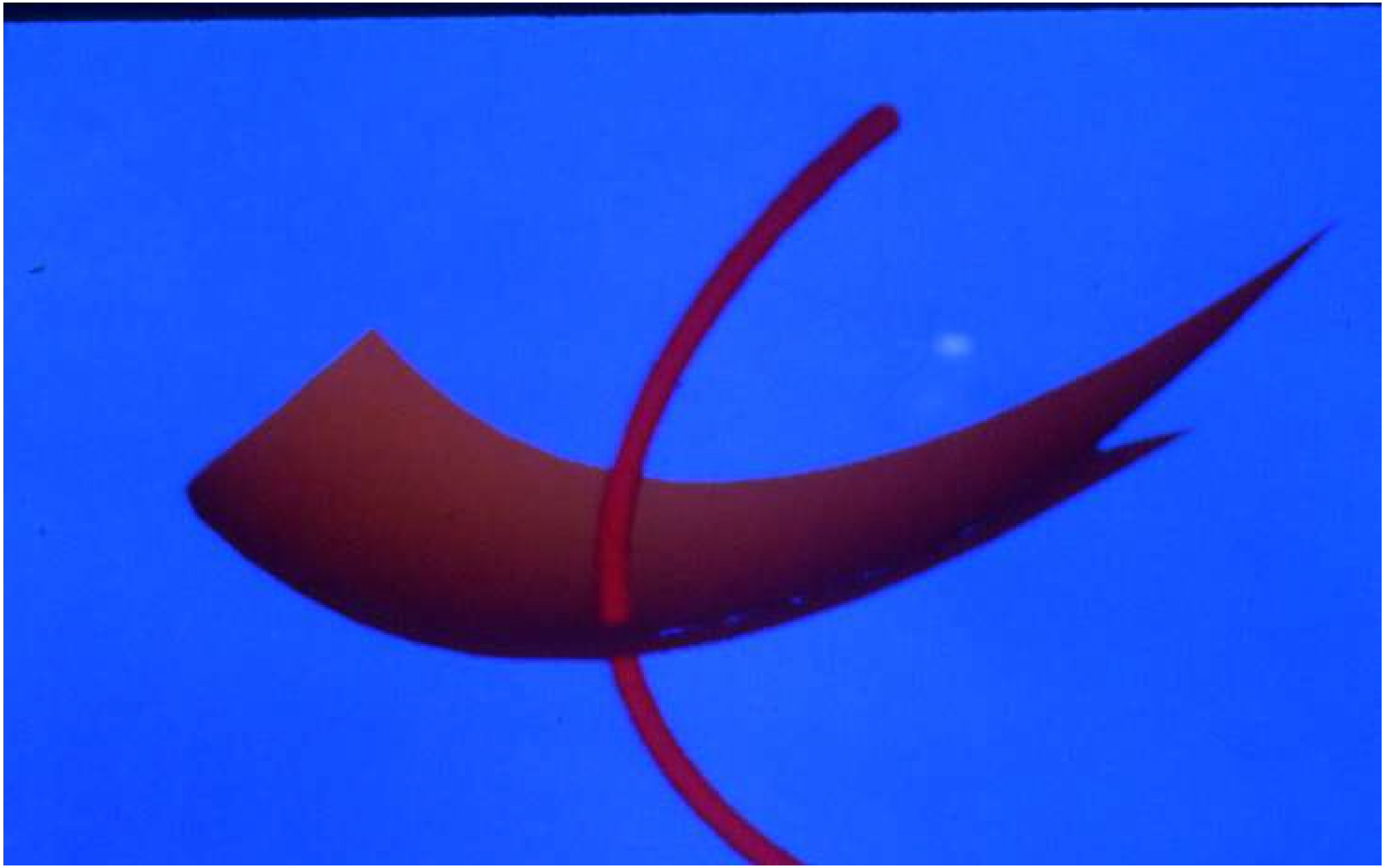
Bezier Clipping

s_{min}	s_{max}	t_{min}	t_{max}
0	1	0	1
.25	.75	.4188	.6303
.3747	.4105	.5121	.5143
.382079	.382079	.512967	. 512967

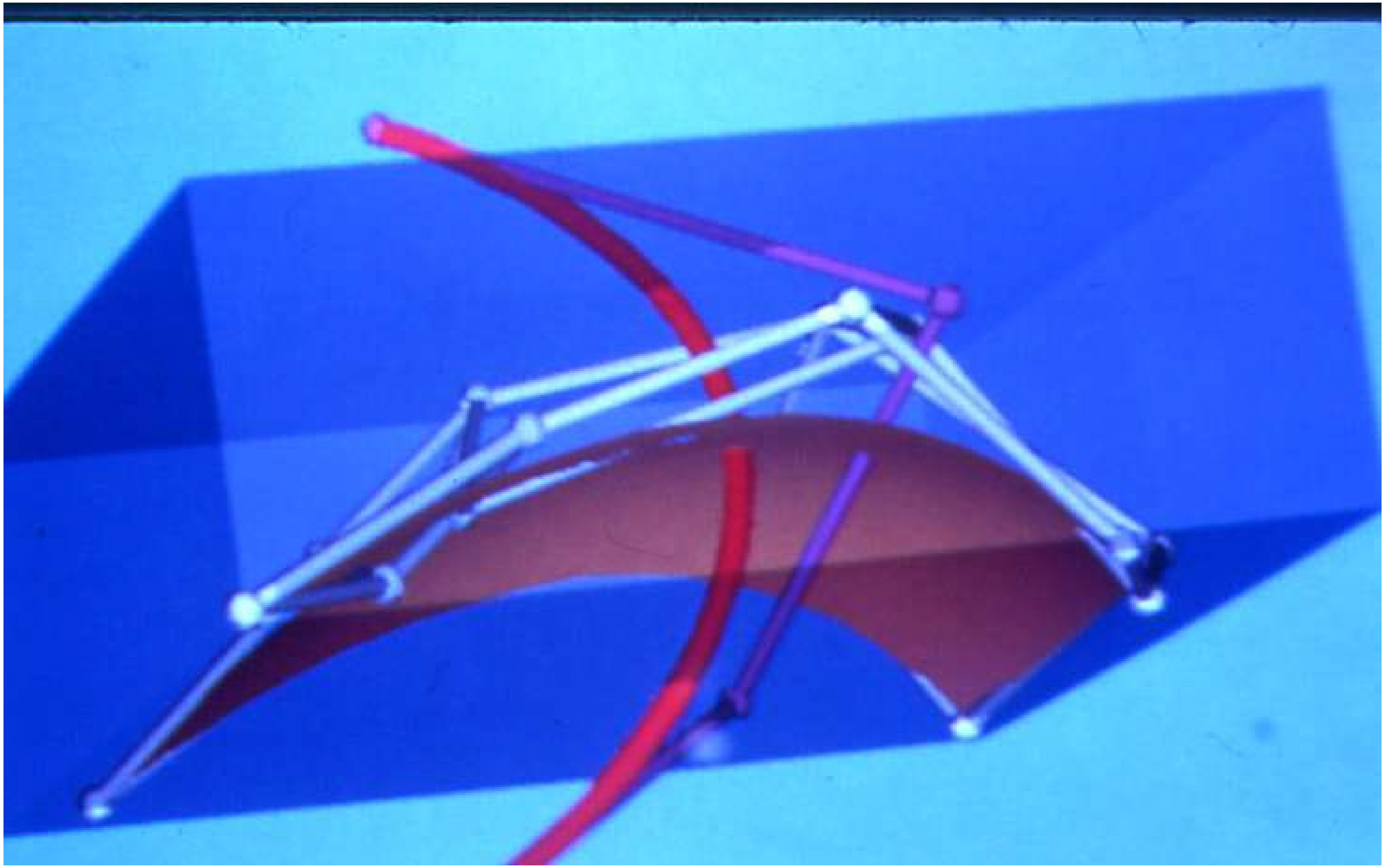
Ray-patch Intersection



Curve-Surface Intersection

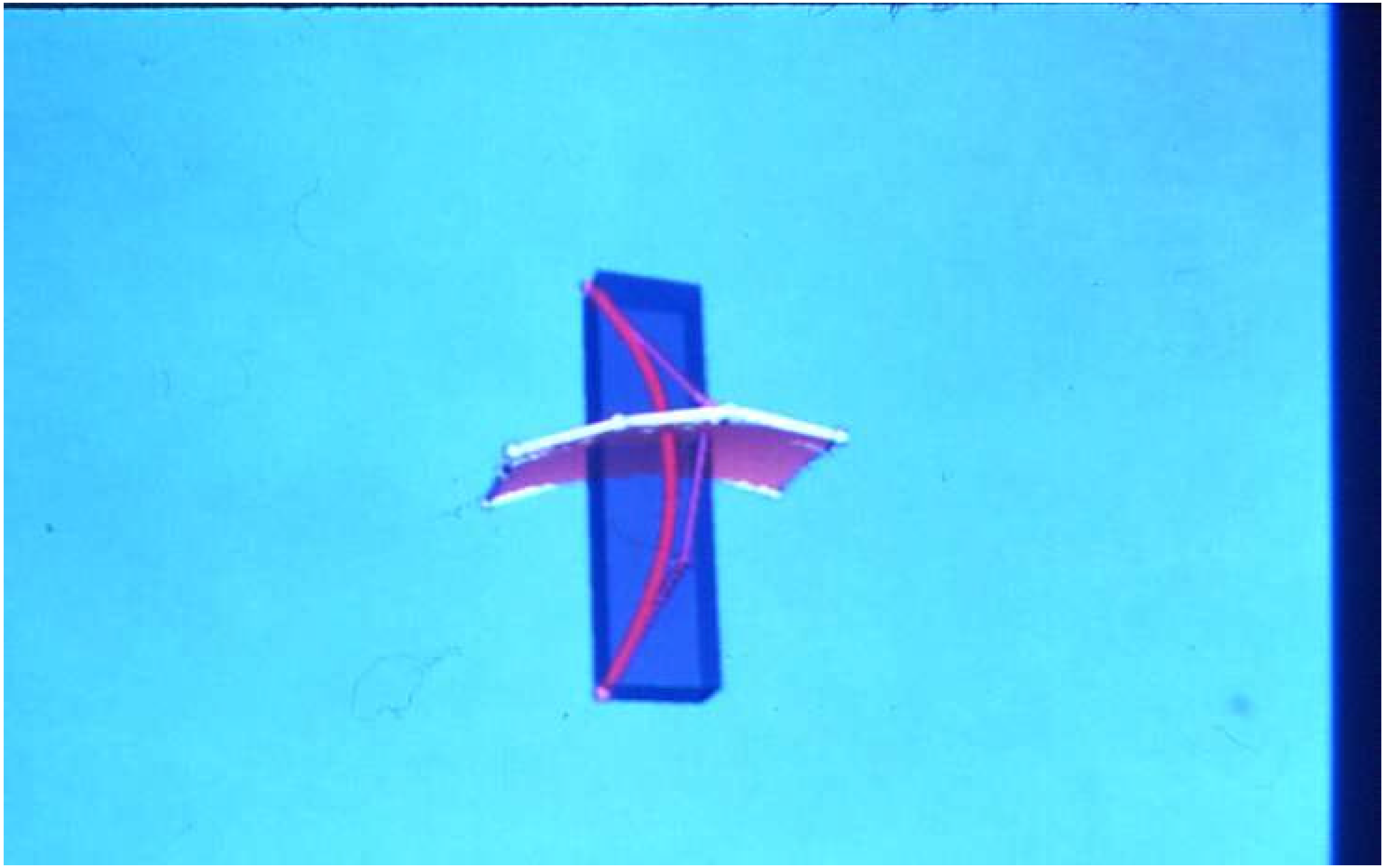


Curve-Surface Intersection

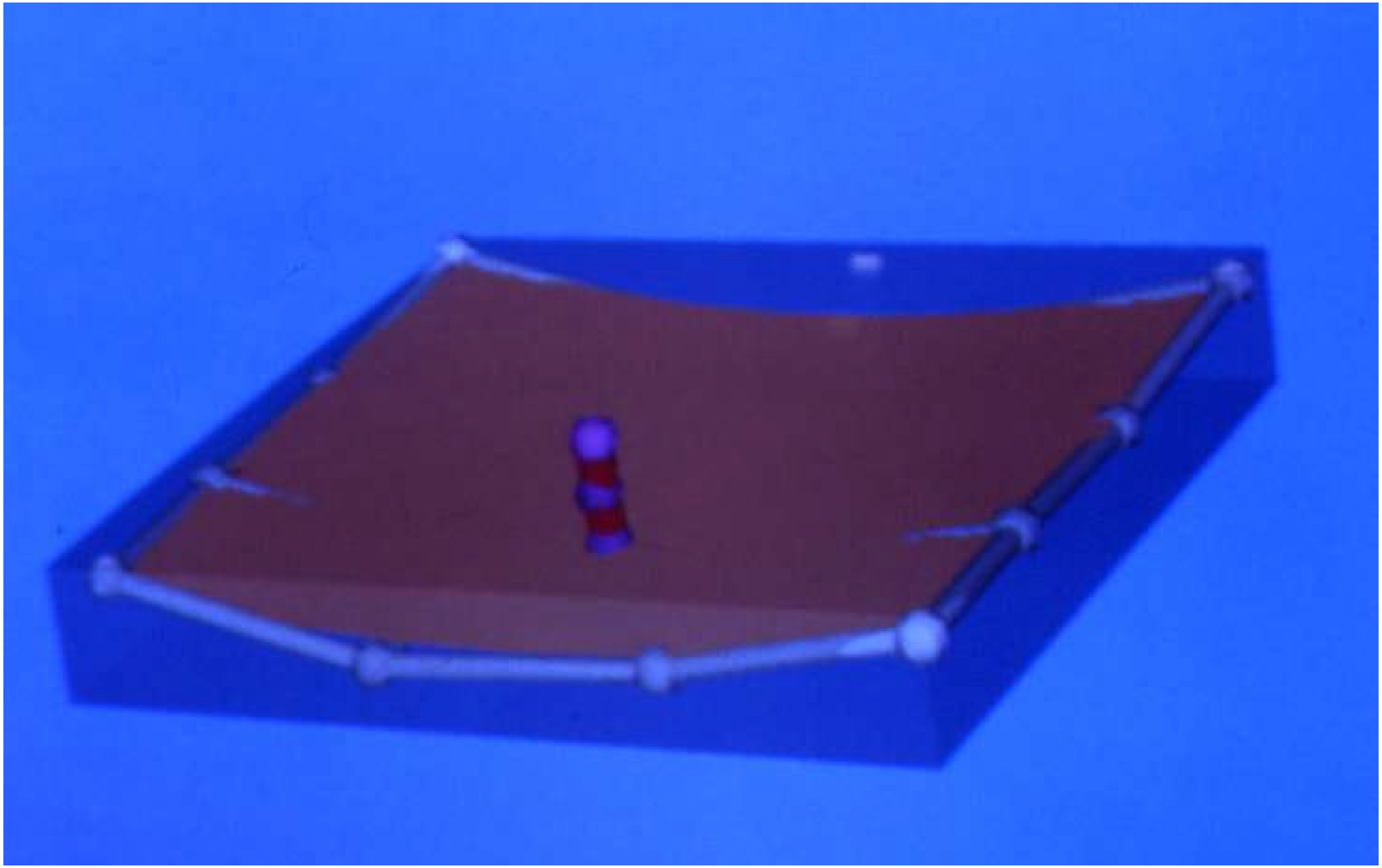


Curve-Surface Intersection

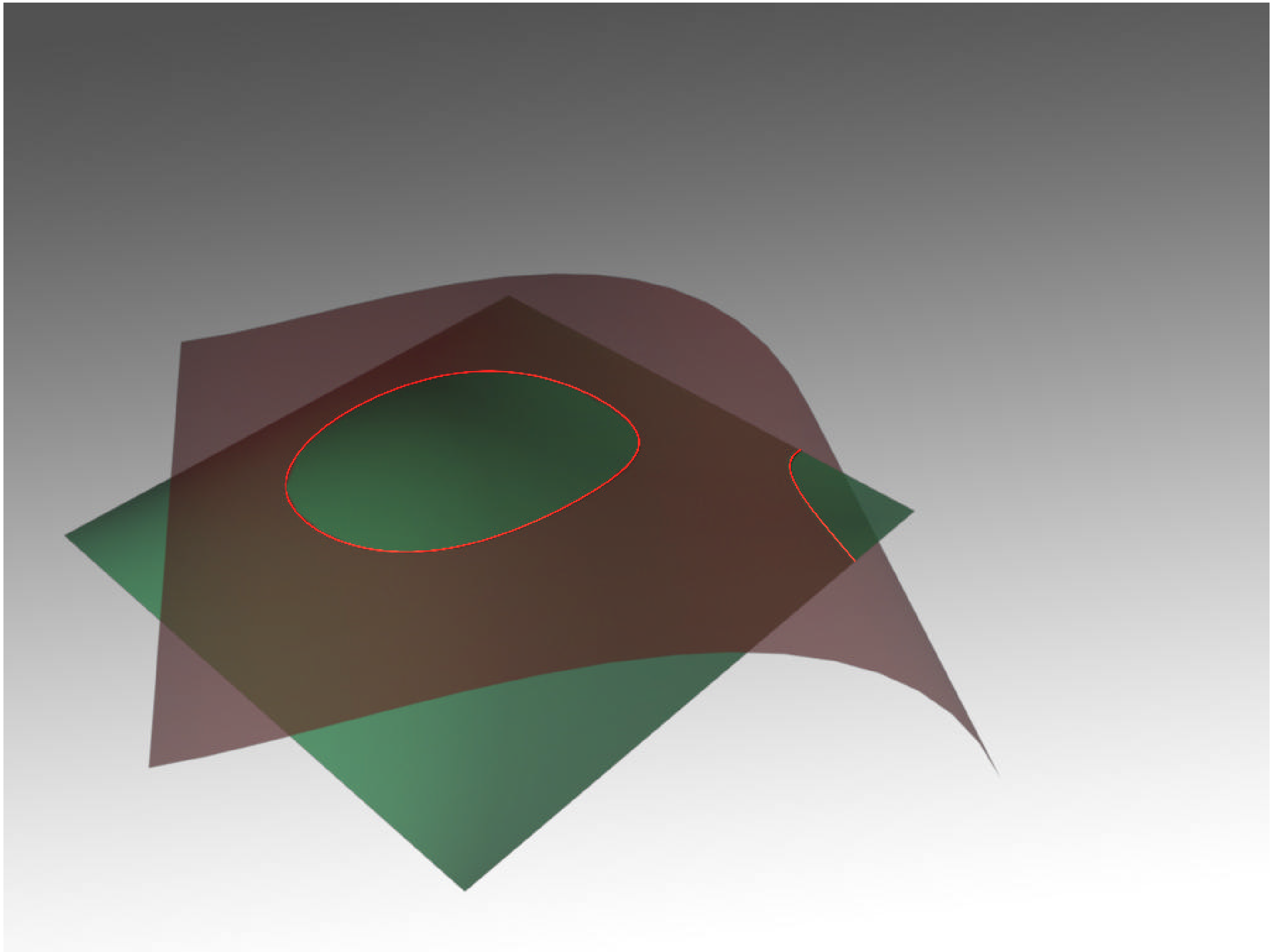
Curve-Surface Intersection



Curve-Surface Intersection



Collinear Normal Theorem



Finding a Collinear Normal Line

$$(P_s(s,t) \times P_t(s,t)) \times (Q_u(u,v) \times Q_v(u,v)) = (0,0,0)$$

$$(P_s(s,t) \times P_t(s,t)) \times (P(s,t) - Q(u,v) = (0,0,0)$$

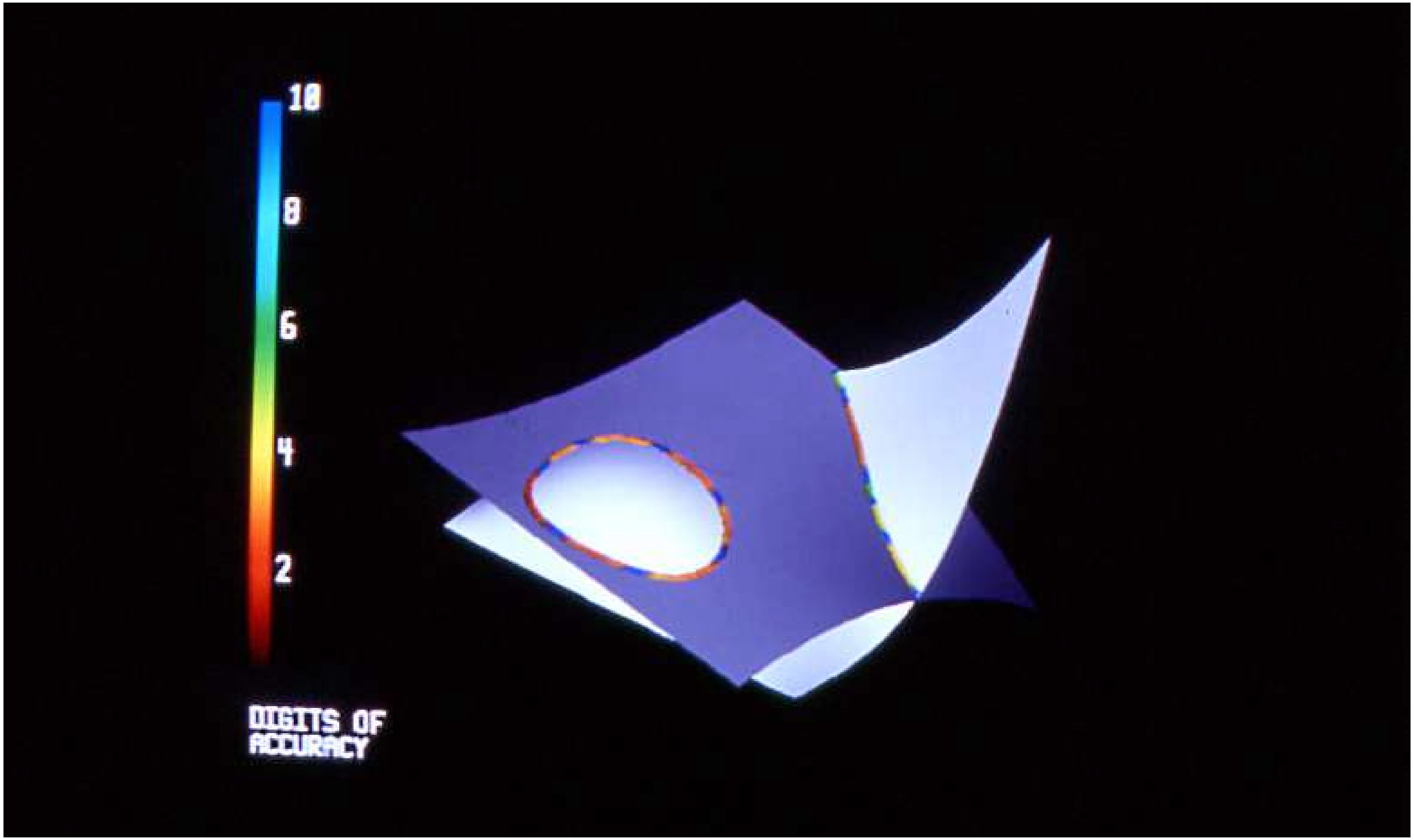
Solutions (s,t,u,v) are parameter values where collinear normal lines pass.

Tangential intersections

Problem 4
Explore the following conjecture

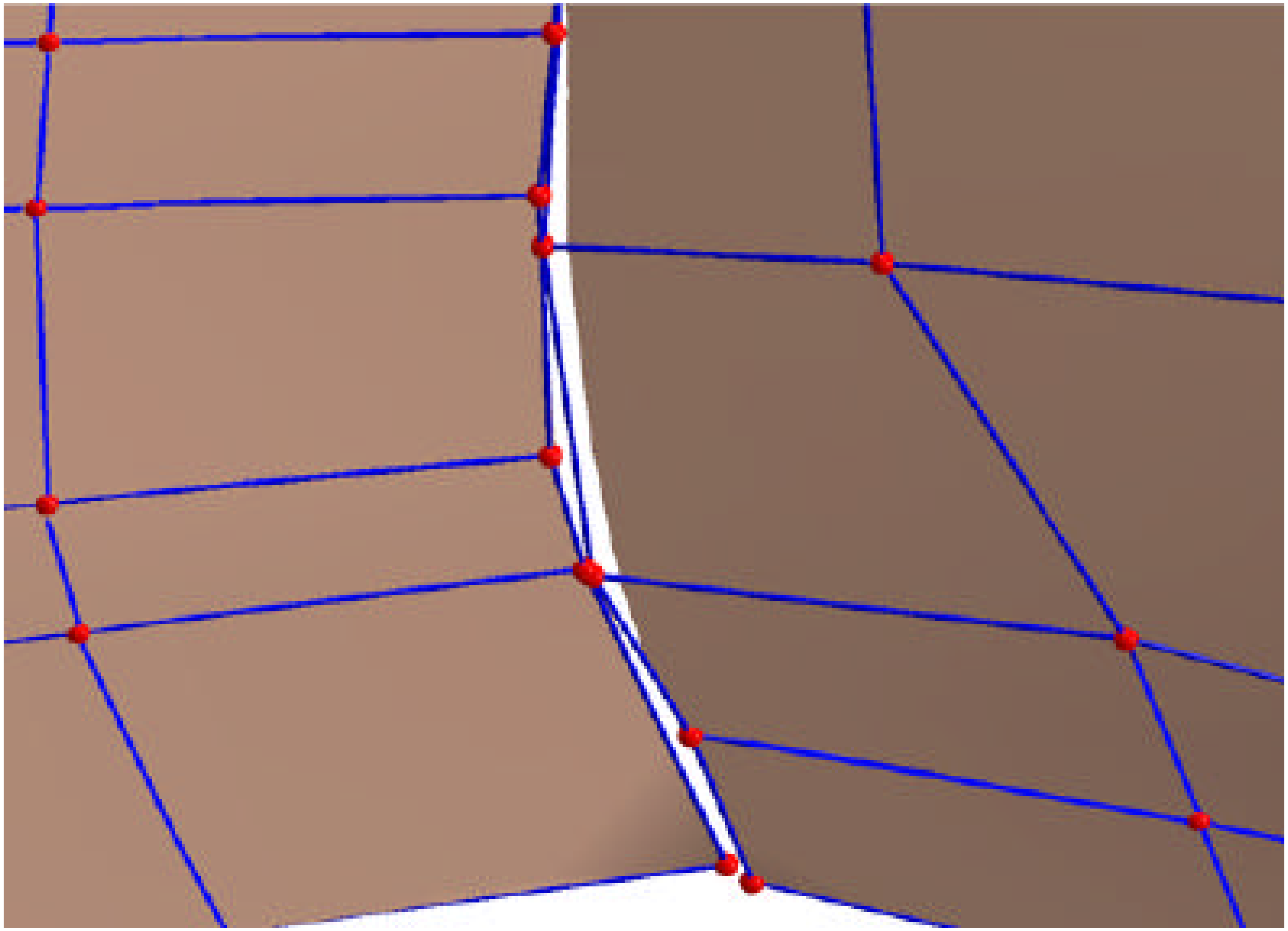
- The probability is infinitesimally small that two bicubic patches will intersect tangentially along a curve that is not parametric (unless the surfaces are intentionally designed to do so)

Problem 5
Filling the Gaps using T-splines

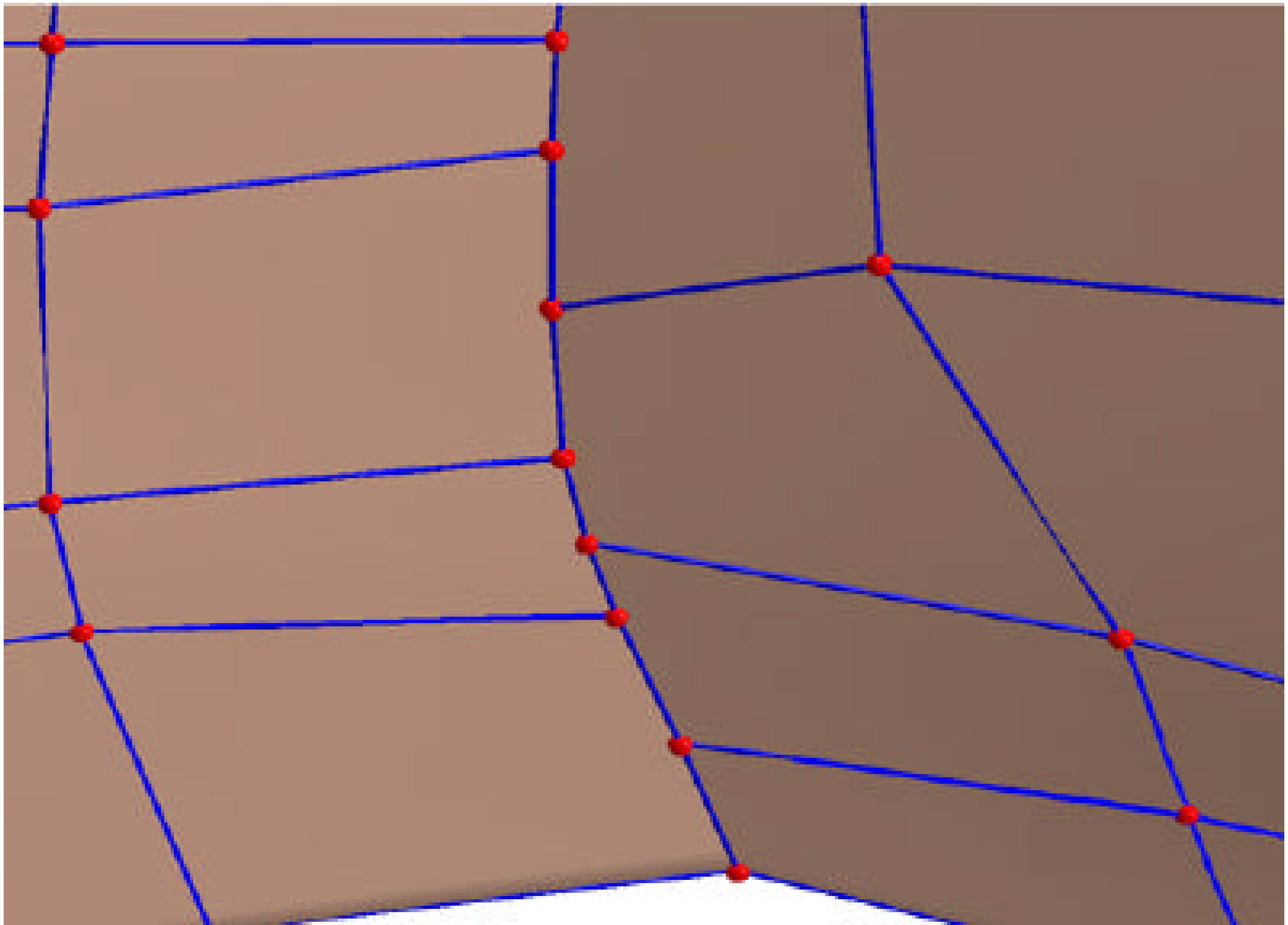


Repairing a B-spline model

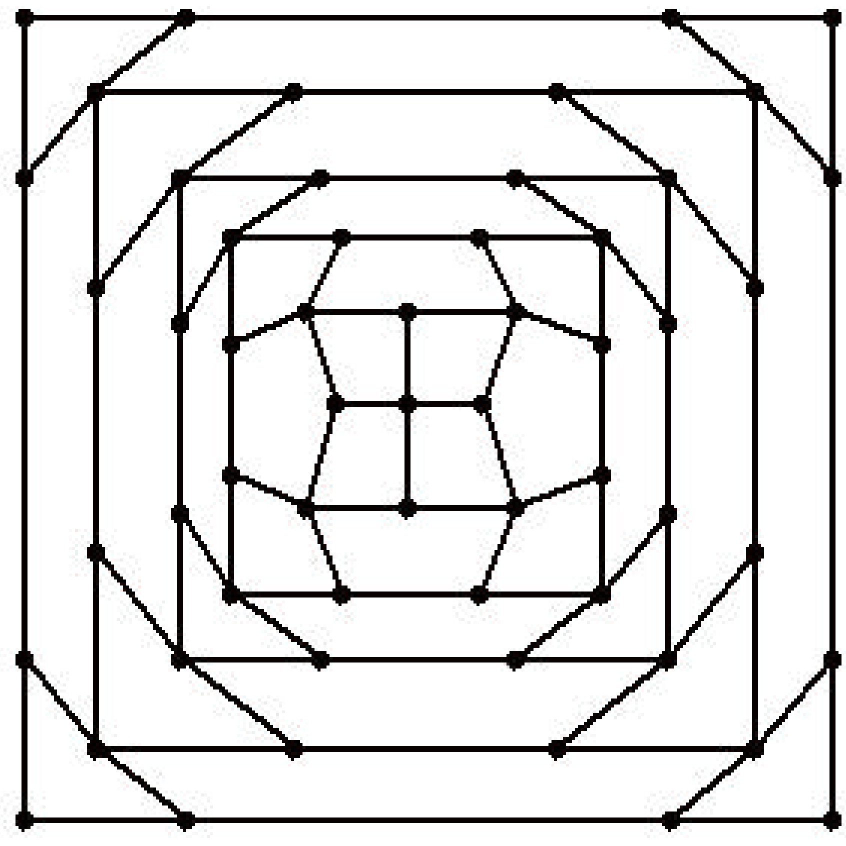
Close-up of Gap



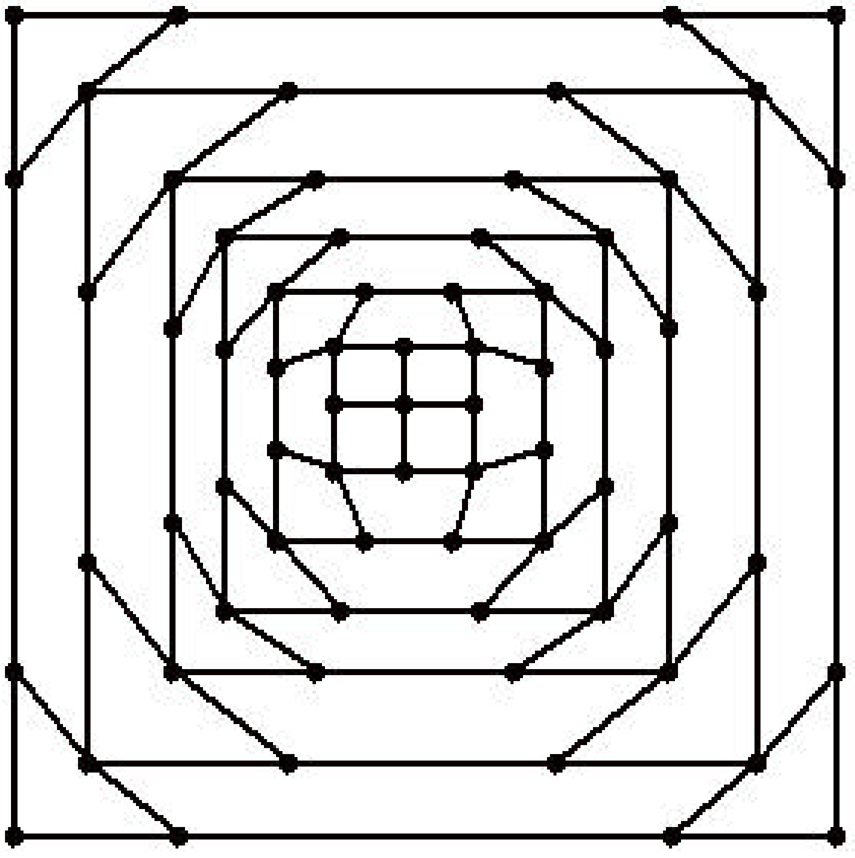
Gap closed using T-splines

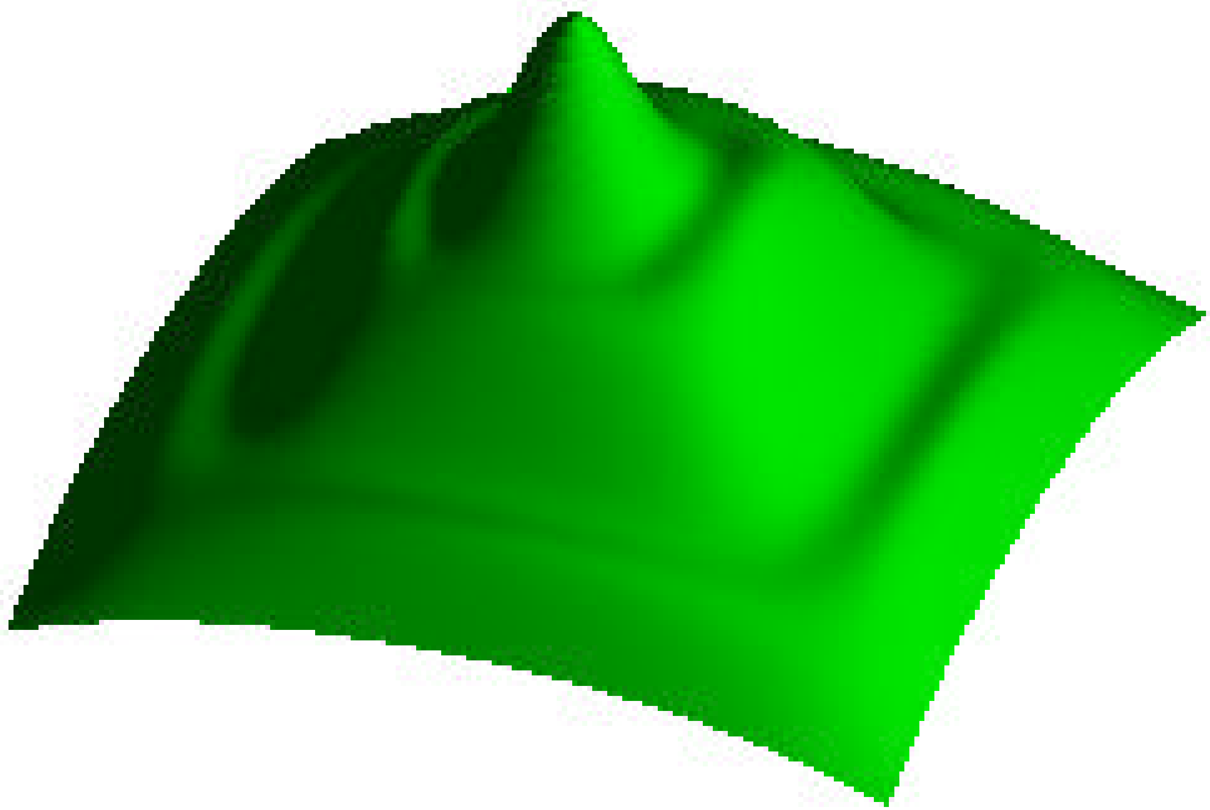


T-spline control grid

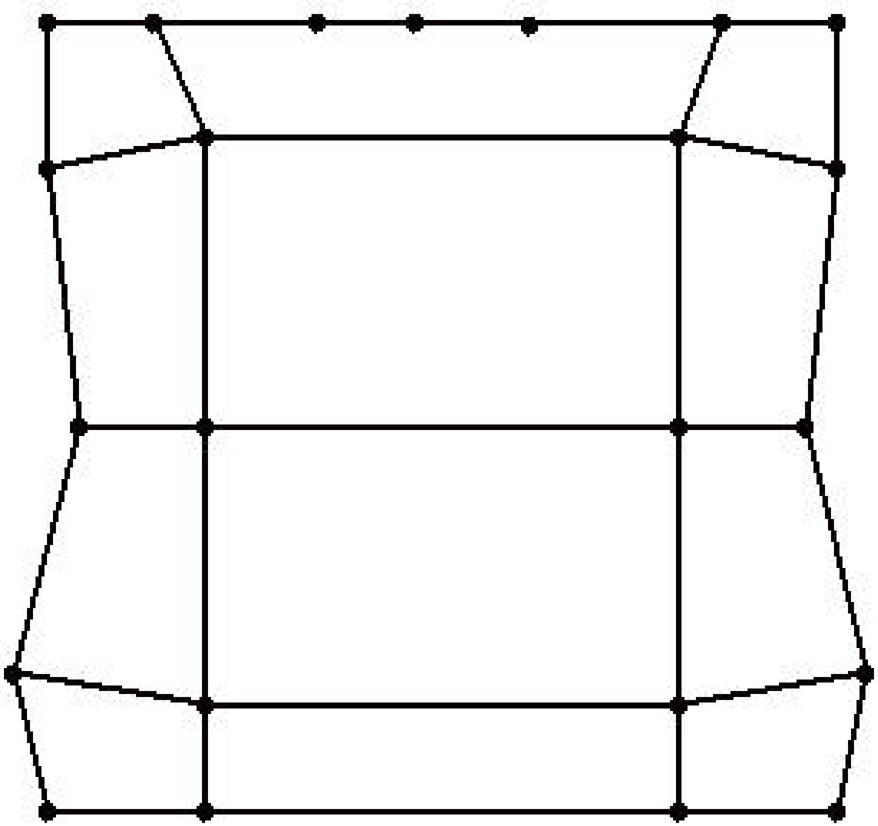


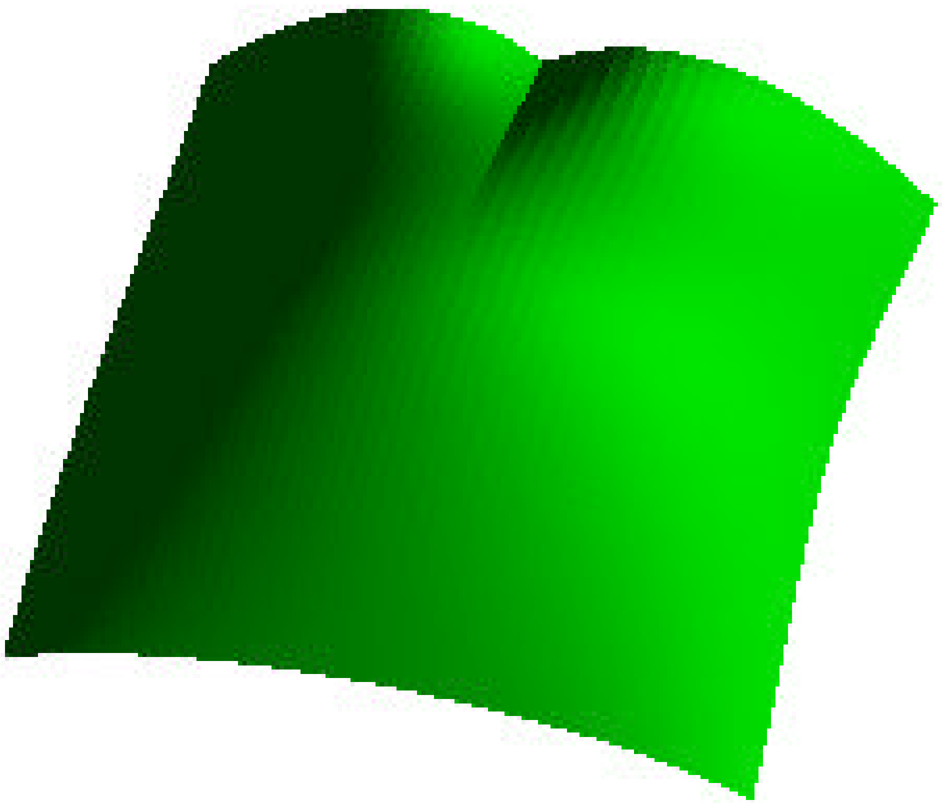
T-spline control grid



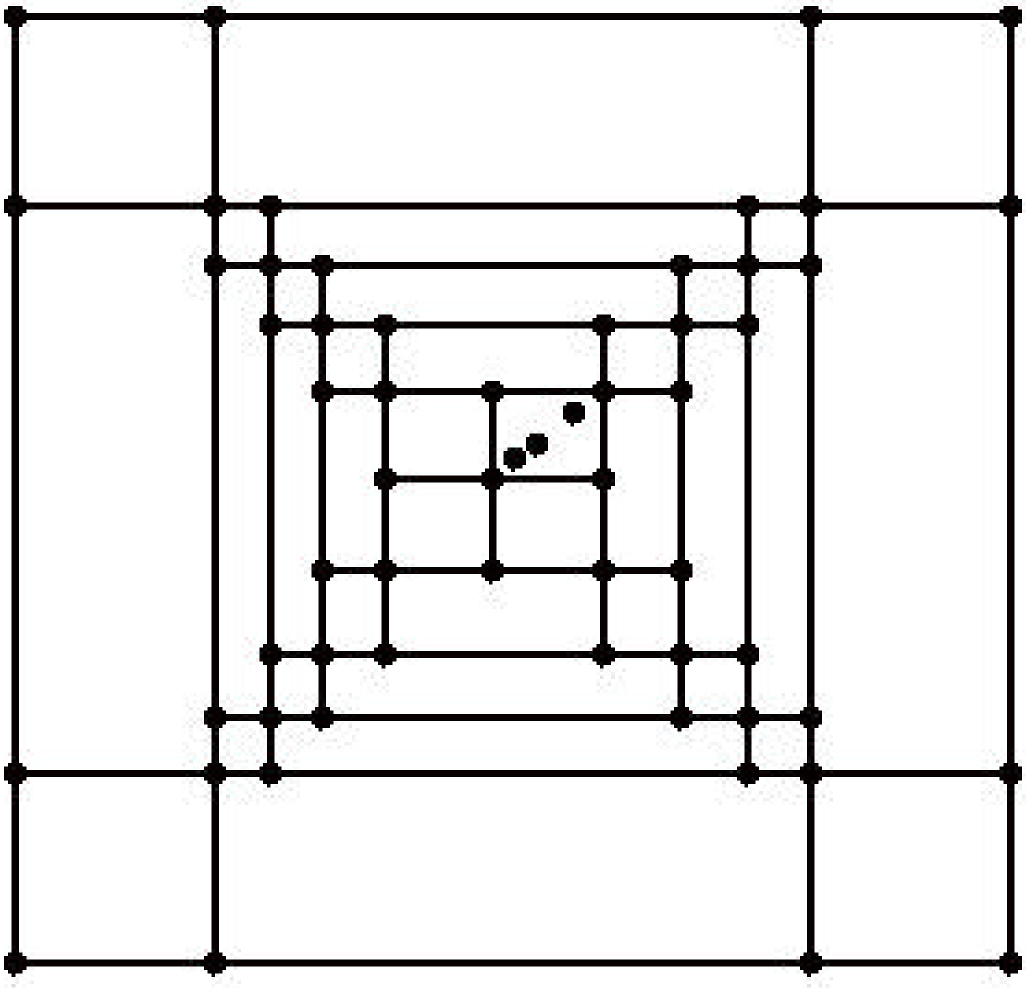


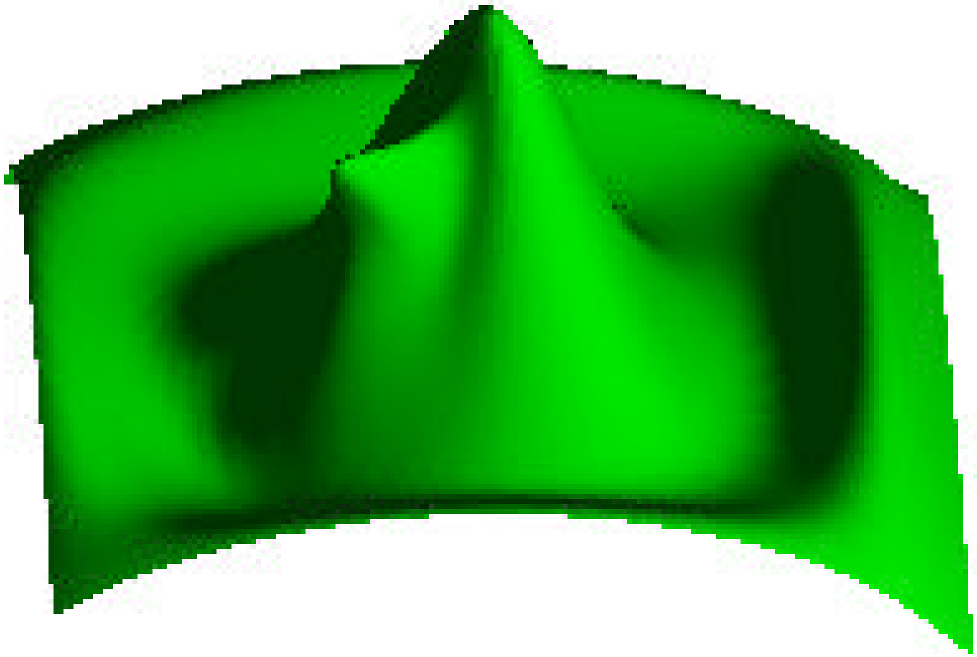
T-spline control grid for Dart



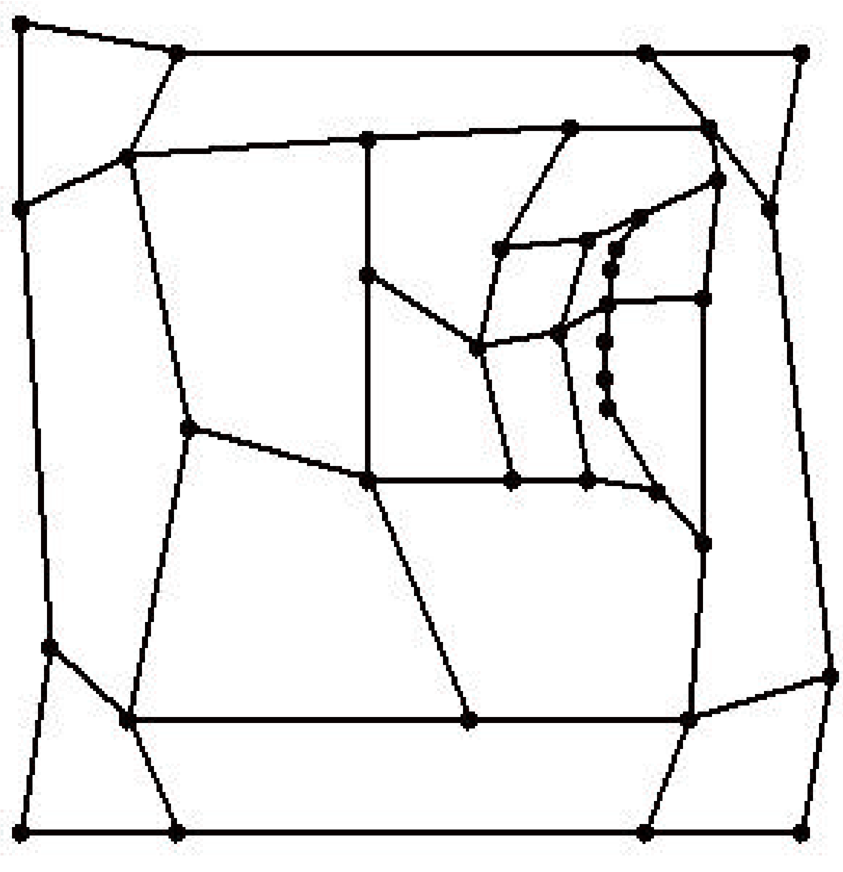


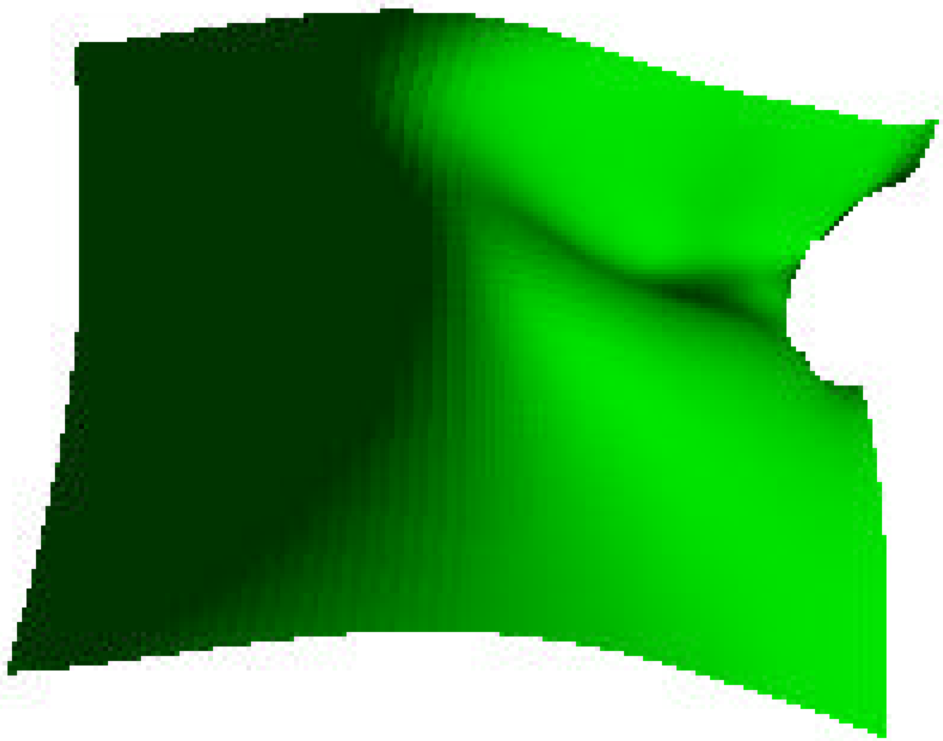
T-spline control grid



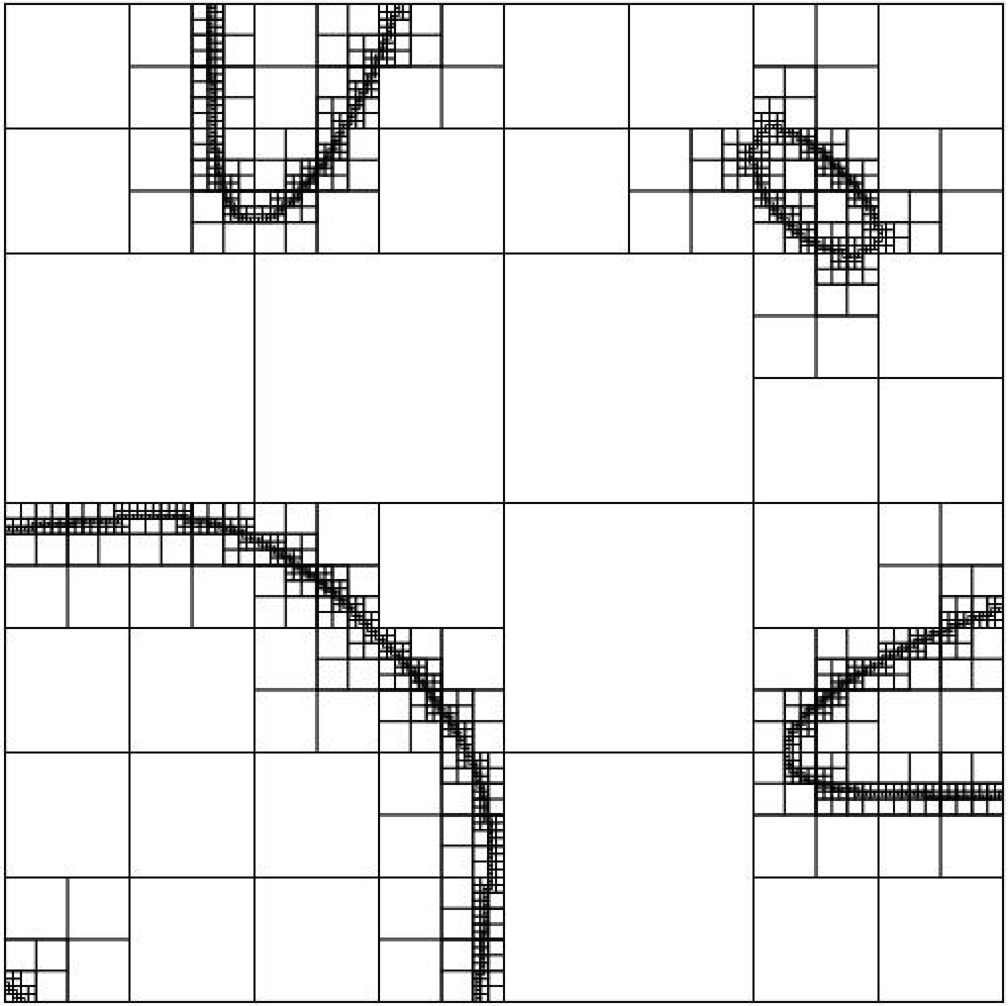


T-spline control grid for trimmed patch

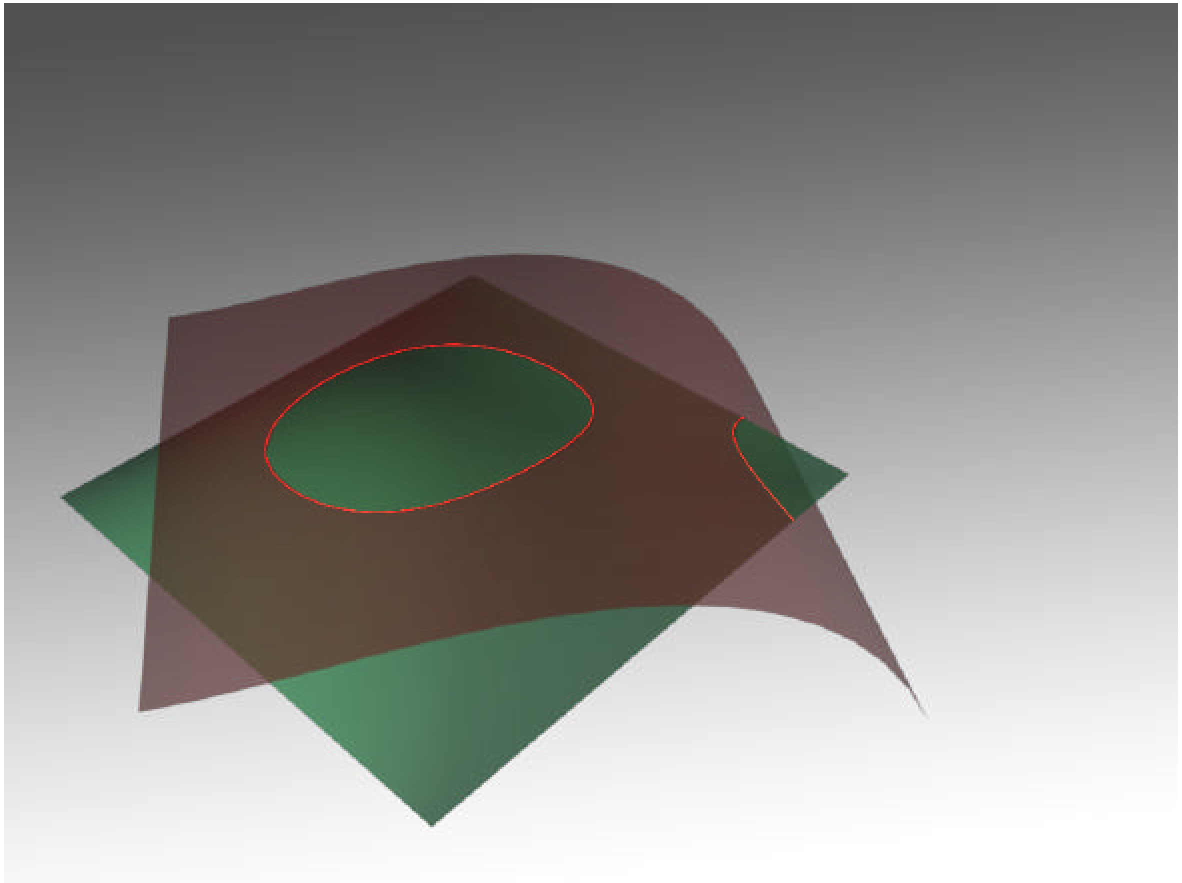


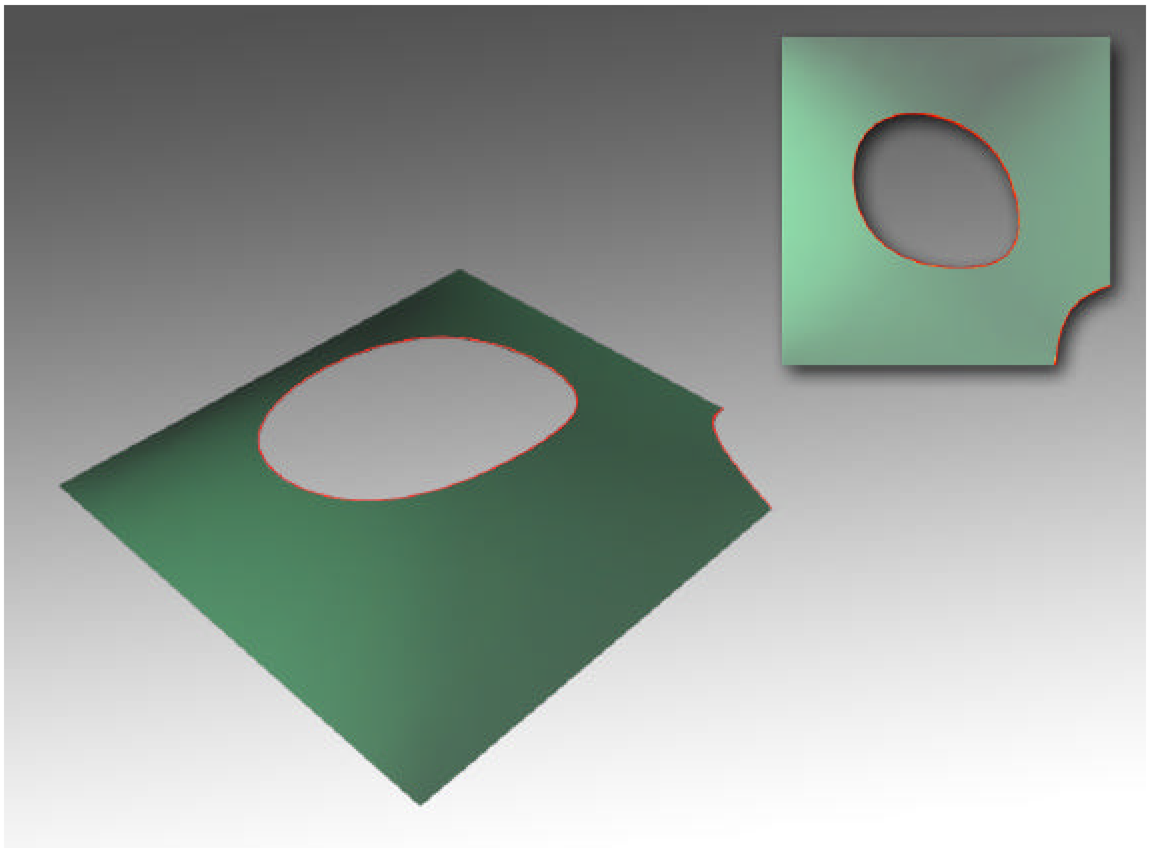


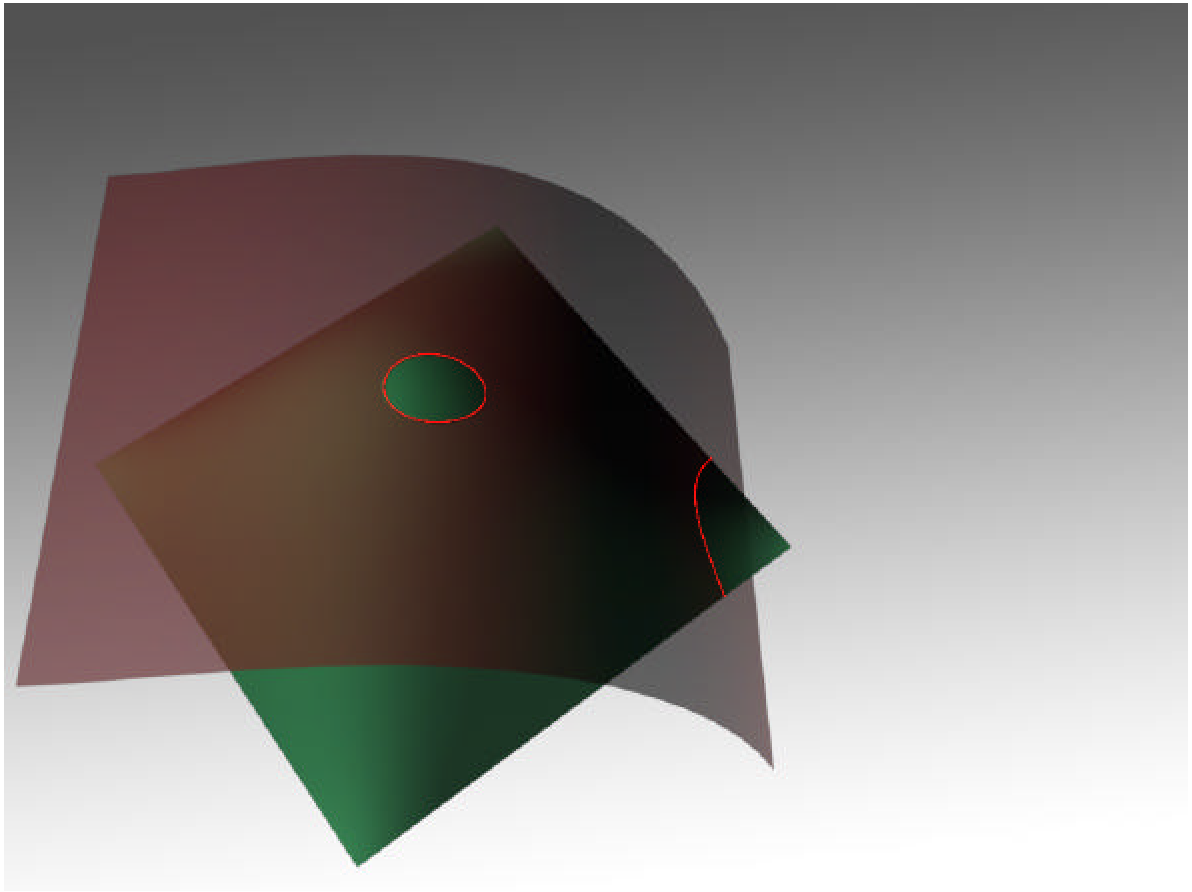
Rough Idea of T-Spline Control Grid for Surface Intersections

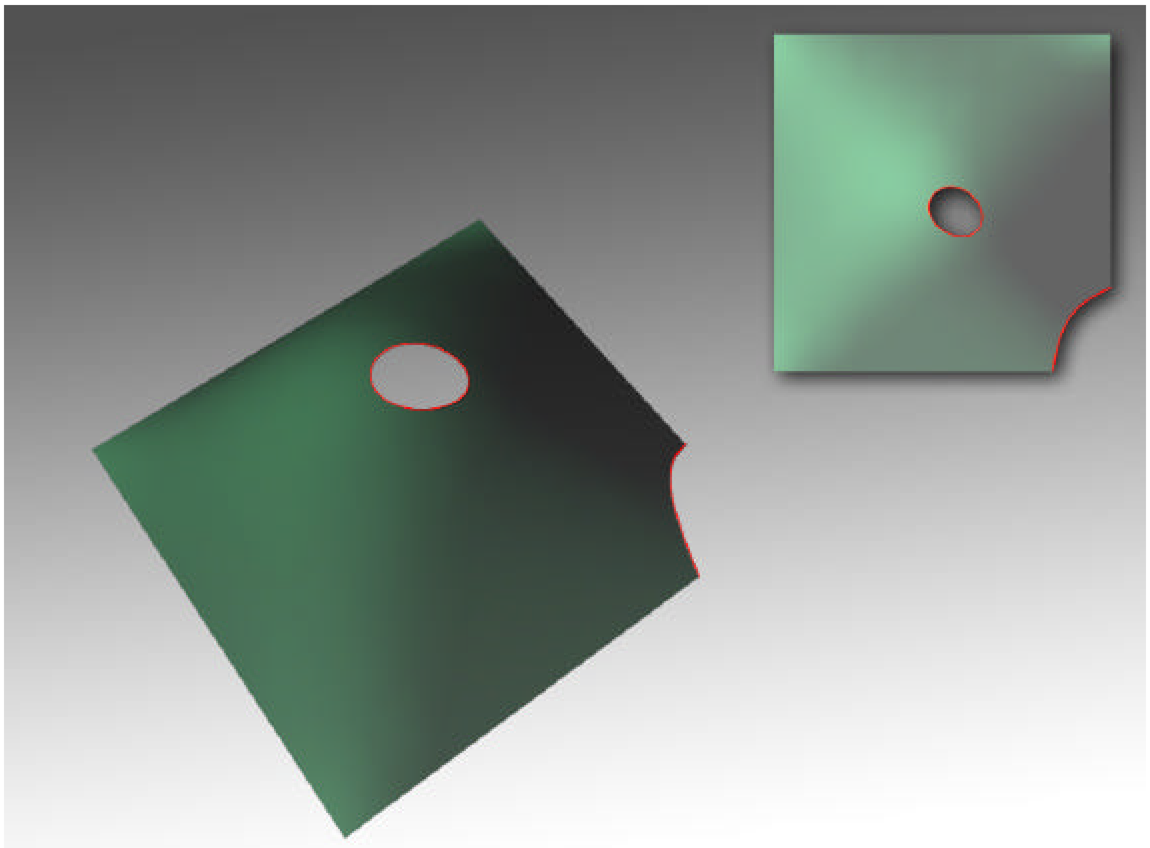


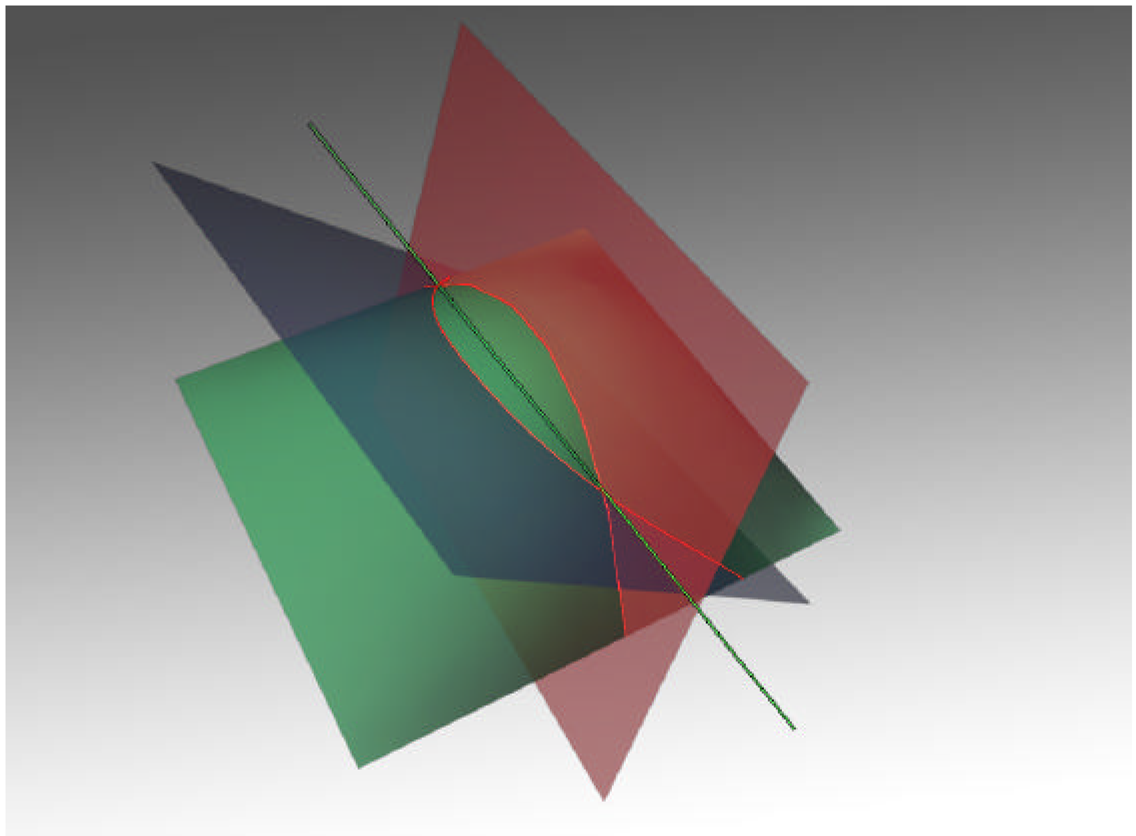
Chee Yap

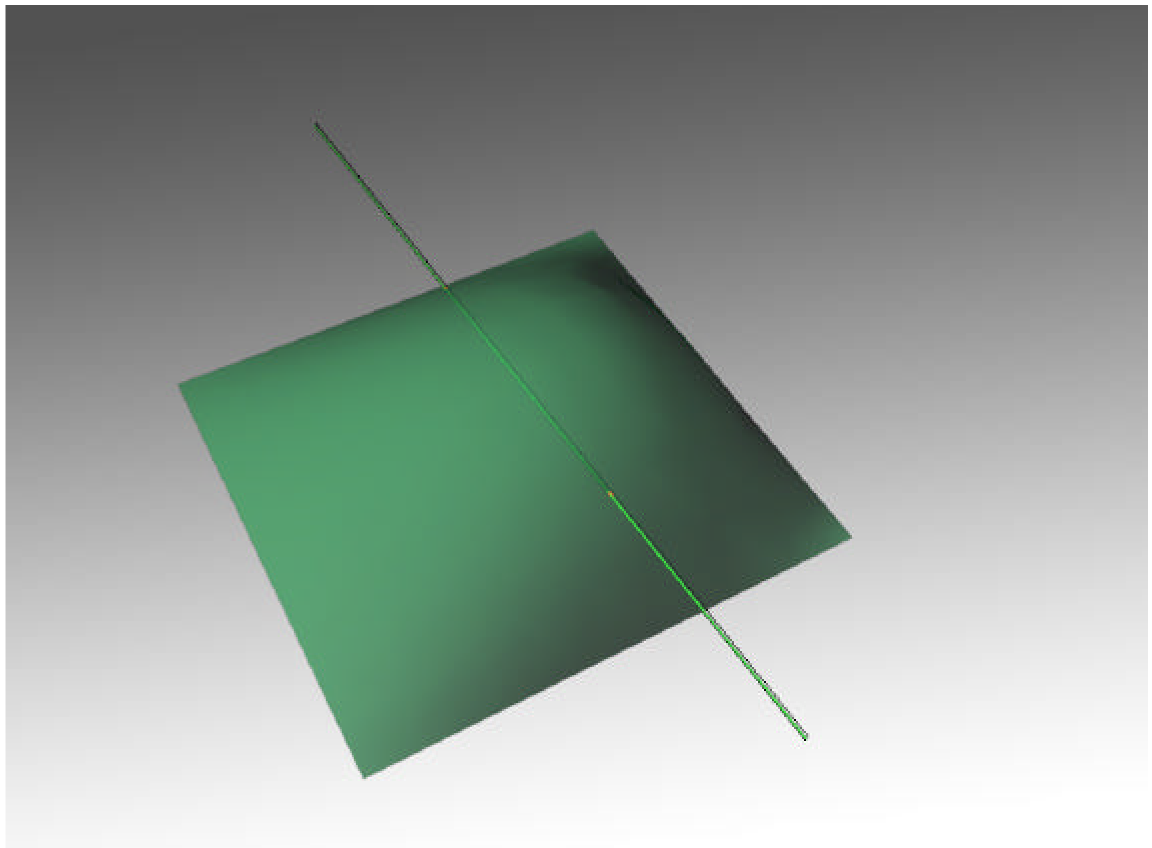


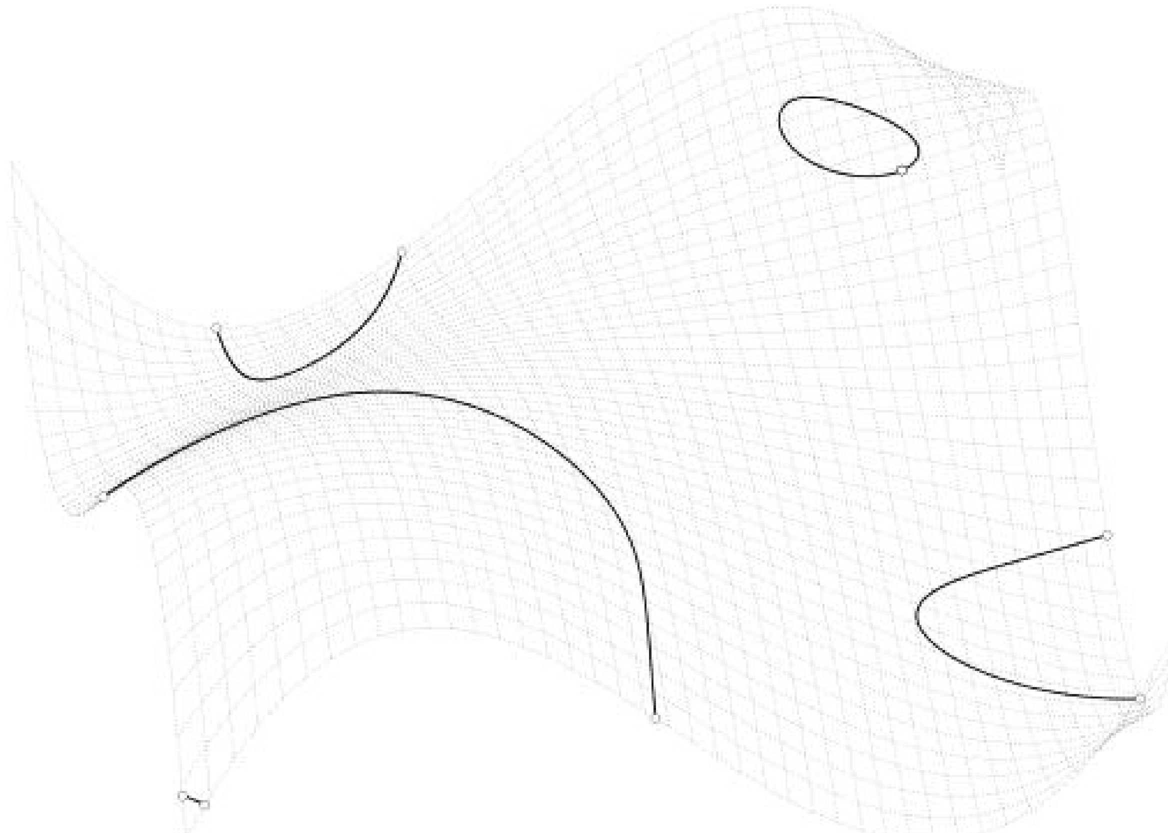












Objectives

- re-visit problems of surface intersections and trimmed surfaces with a fresh perspective
- multi--disciplinary research team, familiar with practical needs of CAD software and the latest mathematical research in geometry and topology.
- Industry suffers exasperation and wasted time using commercial CAD systems, while the CAD research community has largely forsaken fundamental issues and sought refuge in simpler problems leading to easy publications.

- Most surface intersection procedures in current use are based either upon that converge montonically to the surfaces, or numerical tracing procedures coupled with a means of identifying suitable start points. The former approach encompasses {\it subdivision surfaces\/} as well as the B\'ezier/B--spline surfaces. However, it produces only polygonal approximations of the intersection curve in Cartesian space --- information on its behavior in the surface parameter domains, required for surface trimming operations, is missing. The latter approach has the disadvantage of requiring finite--size steps in the tracing procedure, which may incur errors in resolving the topological connectivity. Neither method currently offers a rigorous basis for developing ``water--tight'' trimmed surface

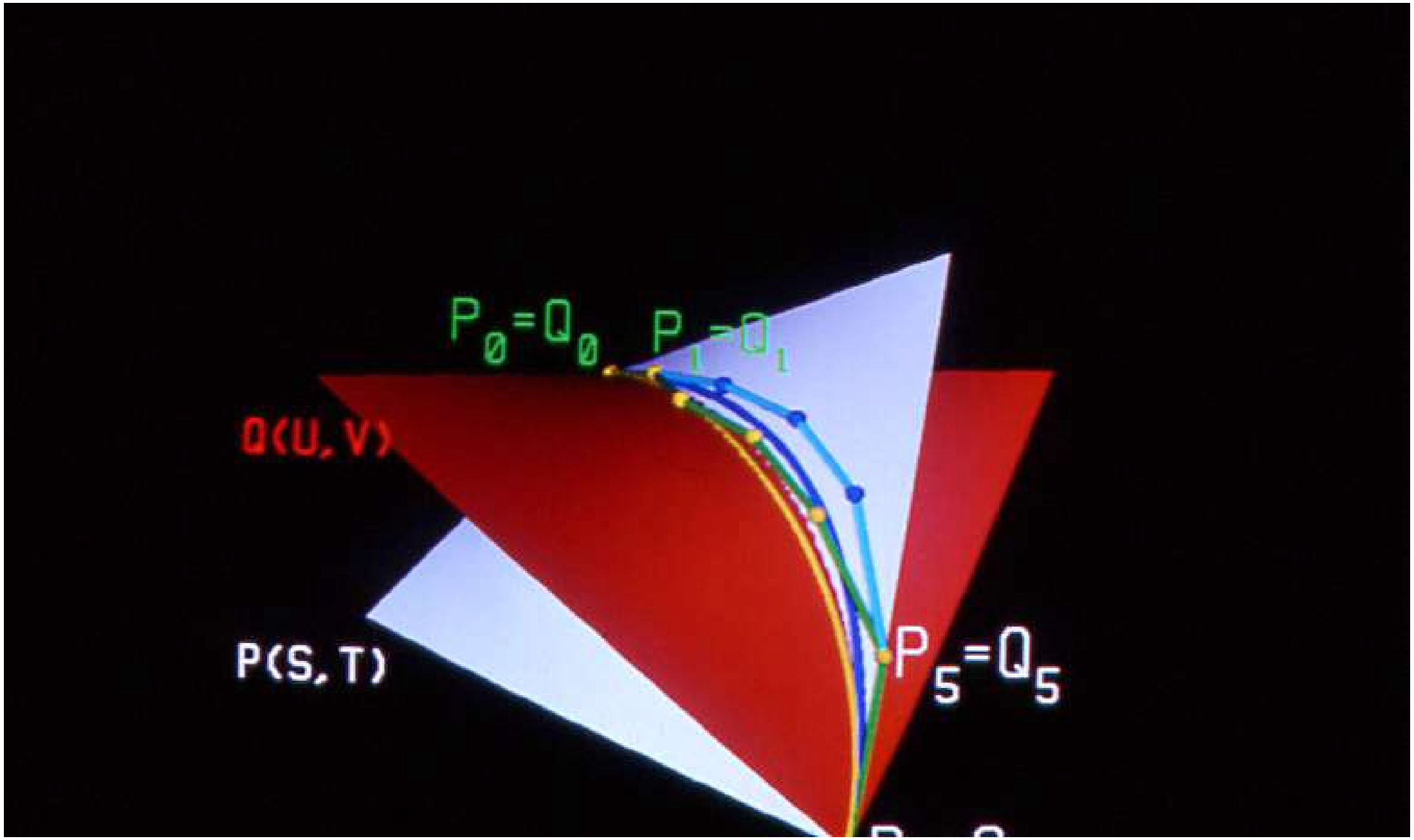
Computational Topology

- UC Davis Mathematics Department
  - strong low-dimensional topology
  - keen interest in computational topology
- Joel Hass
- Jesus De Loera
- Abigail Thompson
- Bill Thurston
- Dmitry Fuchs
- Mikhail Khovanov
- Gregory Kuperberg

Computational Algebraic Geometry

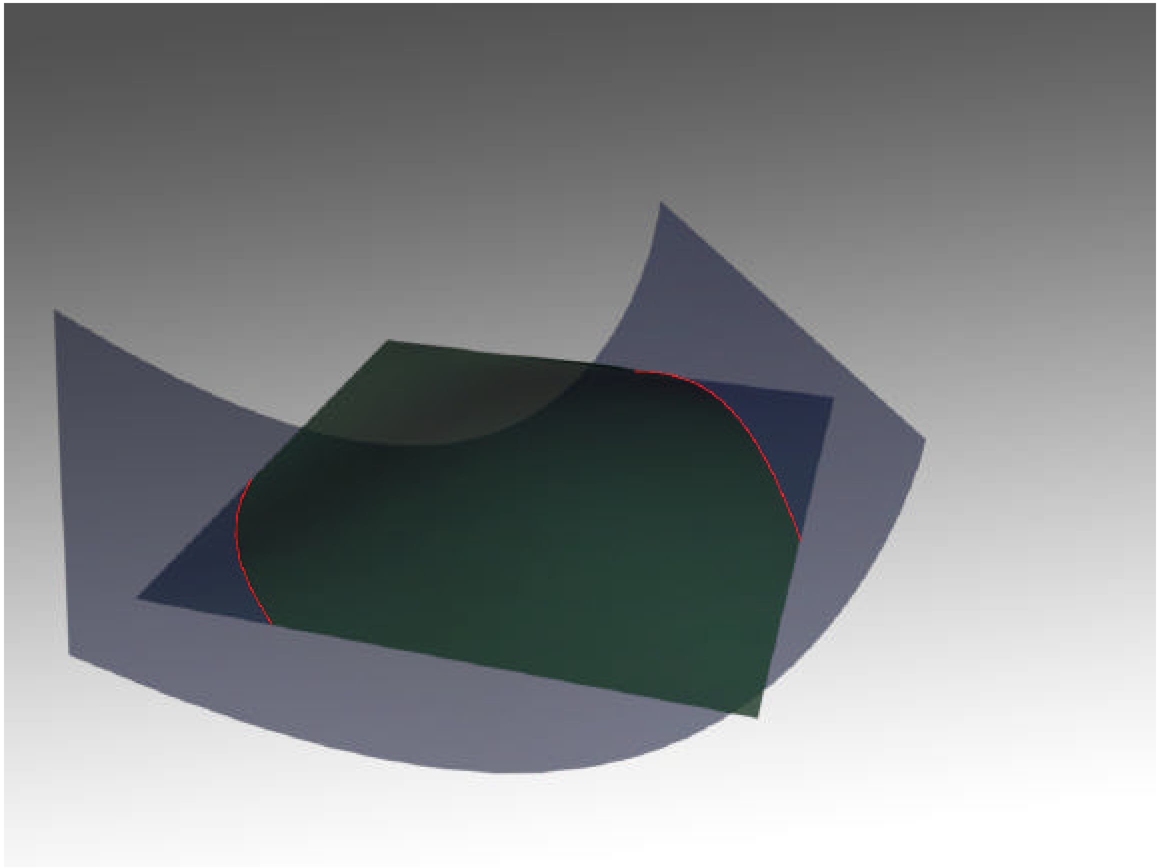
- David Cox, Amherst College
- Ron Goldman, Rice University

- Mesh generation: NASA, Boeing, Ford
  - Geometry preparation and repair takes weeks or months
  - CADFIX reads in CAD model and fixes based on heuristics

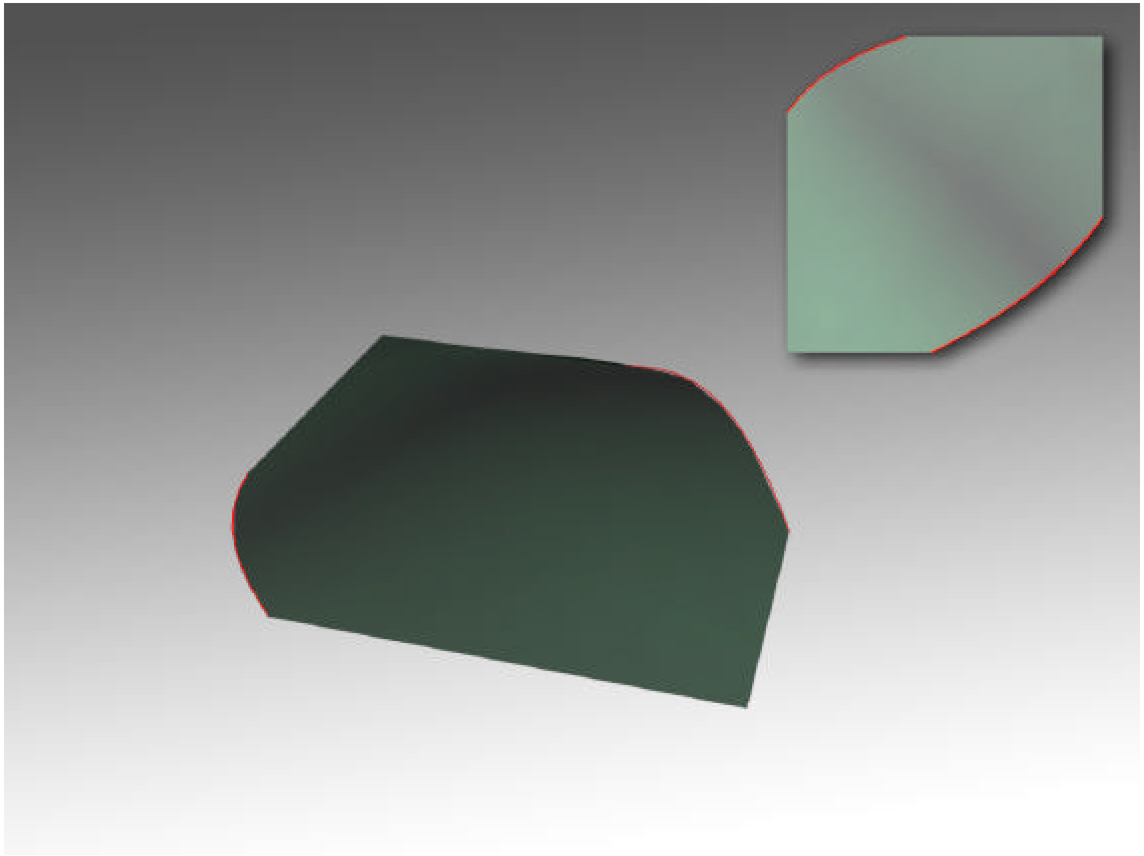


P0=Q0
P1=Q1
Q(U,V)
P(S,T)
P5=Q5

Bi-cubic patch intersections



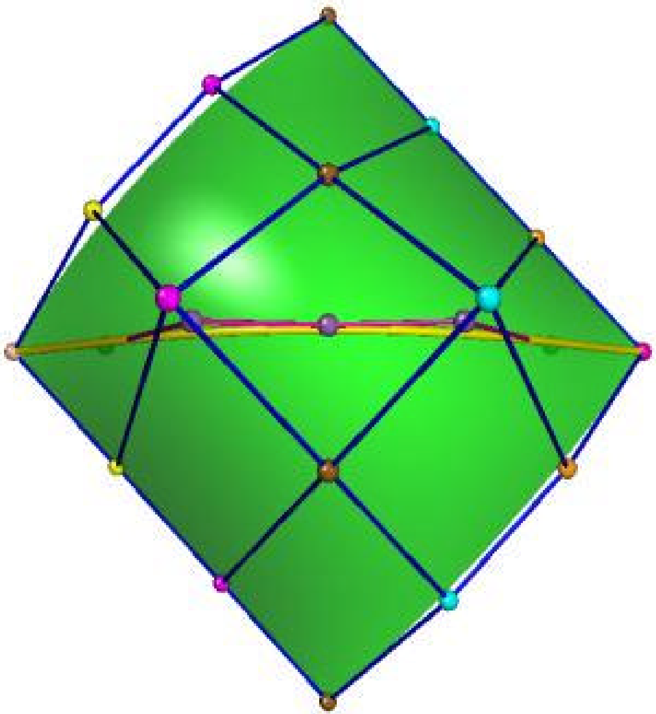
Trimming curves



Performance

- “5%-15% of time spent in performing a Boolean operation using trimmed surface patches is spent fixing gaps.”
- —Tim Strotman, EDS

Diagonal Curve: *s=u, t=u*

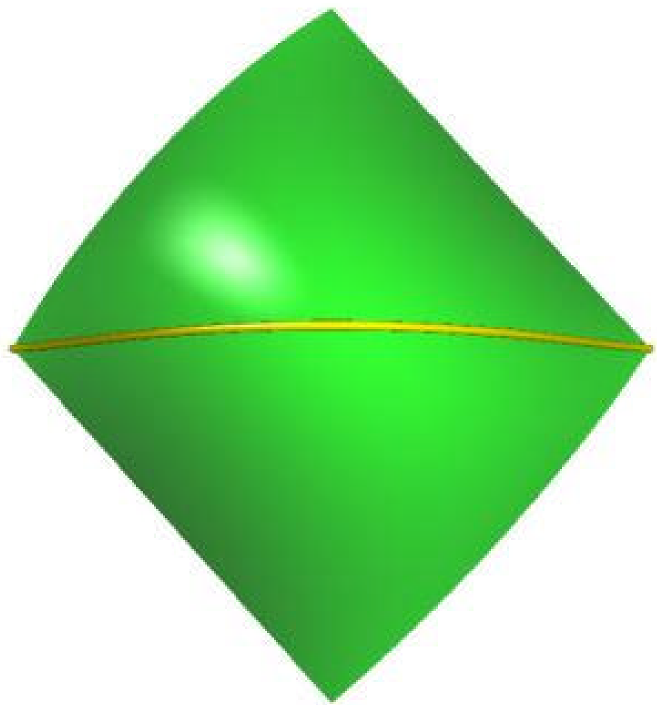


SIAM Workshop on Integration of CAD and CFD (Davis, April 1999)

FE mesh generation from CAD models

- Errors or inconsistencies plague most CAD models
- Primarily due to surface intersection algorithms
- Meshing algorithms are reliable, given “correct” CAD input
- Commercial software (e.g., *CADfix*) detect and fix defects in CAD models, relies on heuristics; no guarantee of success
- Problem occurs in all types of engineering analysis

Diagonal Curve: *s=u, t=u*



Diagonal Curve: *s=u, t=u*